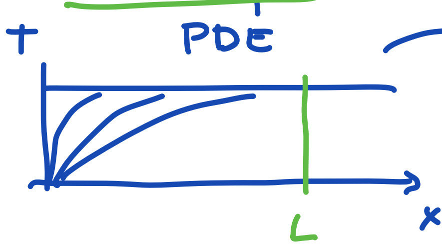
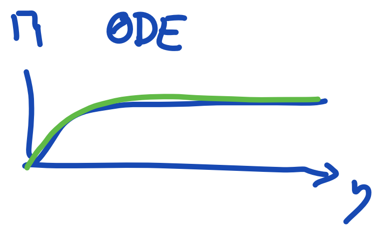
Lecture 20: Advection

Logistics: - HW8 is due (8/11)
⇒ office hours

Last time: - Transient heat conduction
semi-infinite domain





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- similarity variable: $\eta = \frac{x}{\sqrt{4Dt}}$

⇒ $x \sim \sqrt{t}$ heat propagation

Today: - Advection Equation
- Method of characteristics
- Discretization

Advection

Lecture 15: Energy balance

$$\underbrace{\overline{\rho c_p} \frac{\partial T}{\partial t}}_{1} + \nabla \cdot [\underbrace{q \rho_f c_{p,f} T}_{2} - \underbrace{\bar{\kappa} \nabla T]}_{\text{GW flow}} = 0$$

Lectures 16-19: $\underbrace{\rho c_p \frac{\partial T}{\partial t}}_{\text{transient}} - \nabla \cdot \kappa \nabla T = \rho H$

Conduction

If $\overline{\rho c_p}$ = const.

$$\boxed{\frac{\partial T}{\partial t} + \nabla \cdot [\underline{v}_e T - \bar{\alpha} \nabla T] = 0}$$

Advection Diff. Equ.

$$\underline{v}_e = \frac{\phi \rho_f c_f}{\phi \, \overline{\rho c_p}} \underline{v}_f \qquad \underline{q} = \phi \underline{v}_f$$

$$\bar{\alpha} = \frac{\bar{\kappa}}{\overline{\rho c_p}} \quad \text{thermal diffusivity}$$

Scaling: $T' = \frac{T}{T_c}$ $\quad \underline{x}' = \frac{\underline{x}}{x_c}$, $x_c = L$ $\quad t' = \frac{t}{t_c}$, $t_c \equiv t_A$

$$\underline{v}' = \frac{\underline{v}_e}{|\underline{v}_e|}$$

typically T_c and x_c are given (external)

$$\frac{\cancel{T_c}}{t_c}\frac{\partial T'}{\partial t'} + \frac{1}{x_c}\nabla'\cdot\left[\underline{v_c}\,\cancel{T_c}\,T' - \frac{\bar{\alpha}}{x_c}\cancel{T_c}\,\nabla' T'\right] = 0$$

$$\frac{\partial T'}{\partial t'} + \nabla'\cdot\left[\underbrace{\frac{|v_c|\,t_c}{x_c}}_{\underline{\Pi_1}}\,\underline{v}'\,T' - \underbrace{\frac{\bar{\alpha}\,t_c}{x_c^2}}_{\underline{\Pi_2}}\,\underline{\nabla' T'}\right] = 0$$

$$\Pi_1 = \frac{|v_c|\,t_c}{x_c} = 1 \quad \Rightarrow \quad t_c = \frac{x_c}{|v_c|} = t_A$$ advective time scale

$$\Pi_2 = \frac{\bar{\alpha}\,t_c}{x_c^2} = 1 \quad \Rightarrow \quad t_c = \frac{x_c^2}{\bar{\alpha}} = t_D$$ diffusive time scale

$$\left(\frac{L^2}{D}\right)$$

$$x \sim \sqrt{t}$$

choose: $t_c = t_A$

$$\frac{\partial T'}{\partial t'} + \nabla'\cdot\left[\underline{v}'\,T' - \frac{t_A}{t_D}\,\nabla' T'\right]$$

$$\frac{t_A}{t_D} = \frac{x_c\,\bar{\alpha}}{v_c\,x_c^2} = \frac{\bar{\alpha}}{v_c\,x_c} = \frac{1}{Pe}$$

$$\boxed{Pe = \left(\frac{vL}{D}\right) = \frac{v_c\,x_c}{\bar{\alpha}} = \frac{t_D}{t_A}}$$ Peclet number

new notation

$$\frac{\partial T'}{\partial t'} + \nabla' \cdot \left[\underline{v}' T' - \frac{1}{Pe} \nabla' T' \right] = 0$$

Look at $Pe \gg 1$

$$\Rightarrow \quad \frac{\partial T'}{\partial t'} + \nabla' \cdot [\underline{v}' T'] = 0$$

Redimensionality

$$\boxed{\frac{\partial T}{\partial t} + \nabla \cdot [\underline{v}_e T] = 0}$$ Advection equation

Side note : $t_c = t_D$

$$= \frac{\partial T'}{\partial t'} + \nabla' \cdot \left[Pe\, \underline{v}' T' - \nabla' T' \right] = 0$$

$$Pe = \frac{t_D}{t_A}$$

Solution to the Advection equation

Consider heat transport in 1D column with const. $v_e = v = \text{const}$

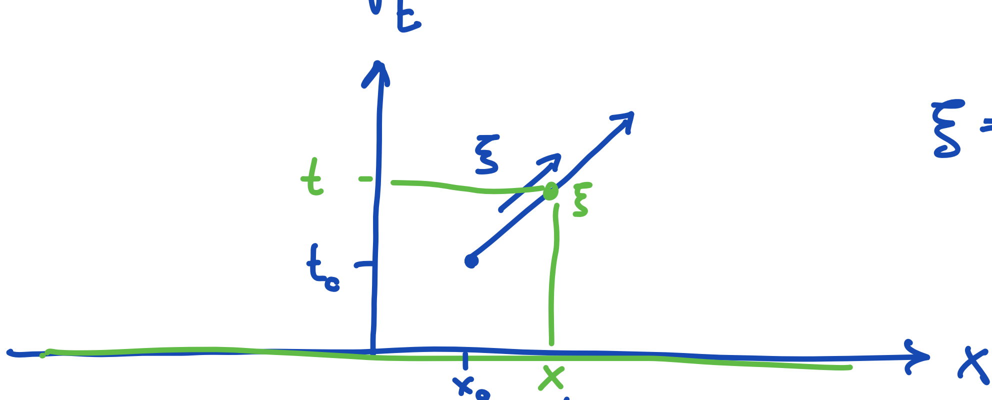
$$\nabla \cdot [\underline{v}_e T] = \underline{v}_e \cdot \nabla T + T (\cancel{\nabla \cdot \underline{v}_e})$$

$$= \underline{v}_e \cdot \nabla T \overset{1D}{=} v_e \frac{\partial T}{\partial x}$$

PDE: $\dfrac{\partial T}{\partial t} + v\,\dfrac{\partial T}{\partial x} = 0 \qquad x \in \mathbb{R}$

IC: $T(x,0) = T_0(x)$

Solve with Method of Characteristics



$\xi = x;$

Idea: Find a characteristic curve, ξ, along which PDE reduces to ODE

$T(x,t) = T\big(x(\xi), t(\xi)\big) = \Theta(\xi)$

Total change in T:

$$\frac{d\Theta}{d\xi} = \frac{d}{d\xi}\Big(T\big(x(\xi), t(\xi)\big)\Big) = \frac{\partial T}{\partial t}\frac{dt}{d\xi} + \frac{\partial T}{\partial x}\frac{dx}{d\xi}$$

compare with PDE: $\dfrac{\partial T}{\partial t} + \dfrac{\partial T}{\partial x}\, v = 0$

1) $\dfrac{d\Theta}{d\xi} = 0$ 2) $\dfrac{dt}{d\xi} = 1$ 3) $\dfrac{dx}{d\xi} = v$

Combine 2 & 3: $\frac{dx}{dt} = v$

Solve for characteristic: $\boxed{x - x_0 = v\,(t - t_0)}$

char. equ

At initial condition:

$$c(x = x_0,\, t = t_0) = c_0(x_0)$$

substitute characteristic $x_0 = x - v\,(t - t_0)$

General analytic solu for Advection equ

$$\boxed{c(x,t) = c_0(x - v\,(t - t_0))}$$

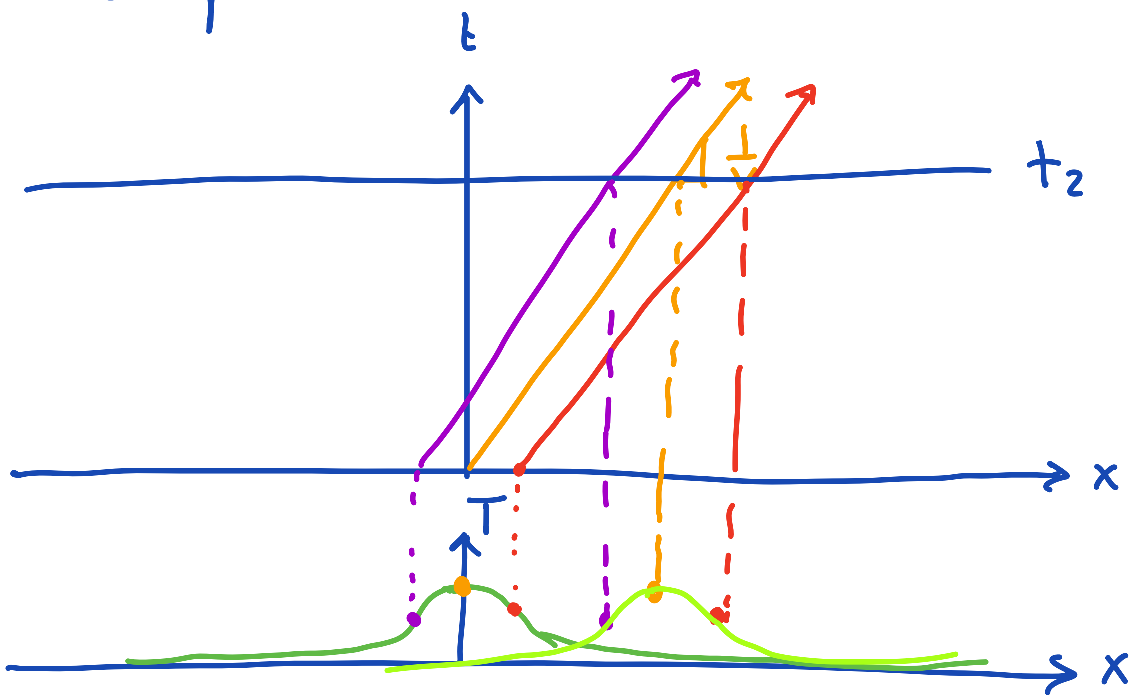
typically $t_0 = 0 \Rightarrow$ $\boxed{c(x,t) = c_0(\underbrace{x - vt}))}$

travelling wave coord.

Definition:

Wave is a signal/disturbance/variation moving through a medium with a recognizable speed of propagation.

$\Rightarrow$ shifts IC to right ($v>0$) without change in shape

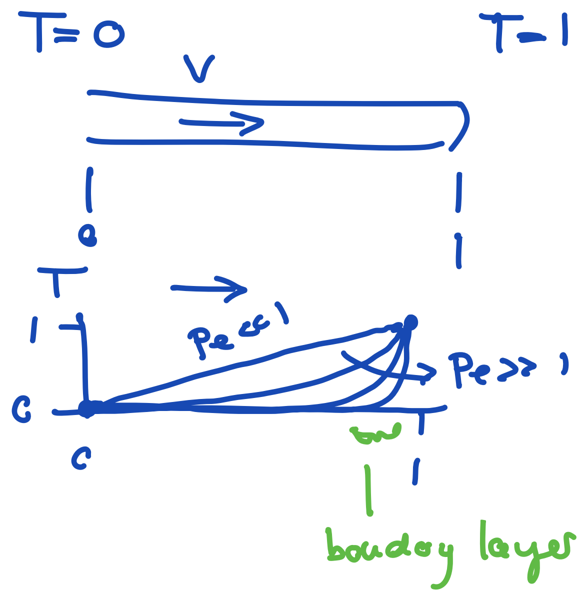


$x - x_0 = v\,t$ $\qquad\qquad$ $t = \frac{1}{v}(x - x_0)$

Steady advection diffusion

Steady flow

PDE: $\frac{d}{dx}\left(\overset{1=v}{Pe}\,T - \frac{dT}{dx}\right) = 0$



BC: $T(0) = 0 \qquad T(1) = 1$

Analytic solution:

$$T(x) = \frac{e^{Pe\,x} - 1}{e^{Pe} - 1}$$

Discretize:

PDE: $\nabla \cdot \left(Pe\,\underline{q}\,T - \nabla T\right) = 0$

Discrete: $\underline{\underline{D}} * \left(Pe\,\underline{\underline{A}}(\underline{q})\,\underline{u} - \underline{\underline{G}}\,\underline{u}\right) = 0$

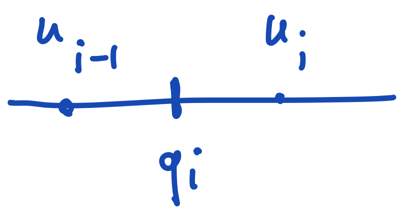
$$\underline{\underline{L}}\,\underline{u} = 0$$

$$\underline{\underline{L}} = \underline{\underline{D}} * \left[Pe\,\underline{\underline{A}}(\underline{q}) - \underline{\underline{G}}\right]$$

$\underline{\underline{A}}$ is $Nx+1$ by Nx matrix that is function of known fluxes $\underline{q}$

$\underline{\underline{A}}(q)\,\underline{u} \approx \underline{q}\,T$

Problem is that $\underline{u}$ is given on cell centers but flux needs to be on cell faces.



Advective flux across i-th interface

$$\boxed{a_i = q_i\, u_{i-\frac{1}{2}}}$$

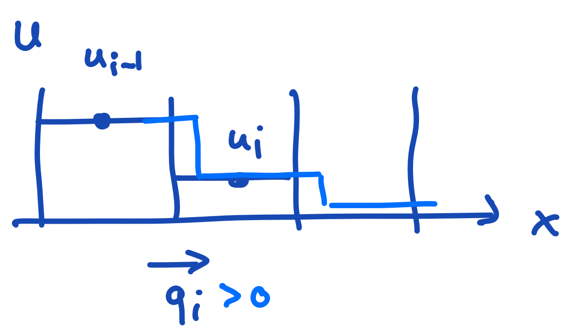
Need to approximate $u_{i-\frac{1}{2}}$

Central flux

$$u_{i-\frac{1}{2}} = \frac{1}{2}(u_i + u_{i-1})$$

$\Rightarrow$ leads to oscillations

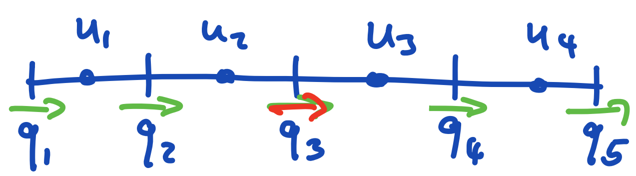
Upwind flux



from MOC we know that solution is simply shifted to the right $(q \geq 0)$. $\Rightarrow$ $u_{i-\frac{1}{2}} =$

$$\boxed{u_{i-\frac{1}{2}} = \begin{cases} u_{i-1} & \text{if} \quad q_i > 0 \\ u_i & \text{if} \quad q_i < 0 \end{cases}}$$

What is $\underline{\underline{A}}$:



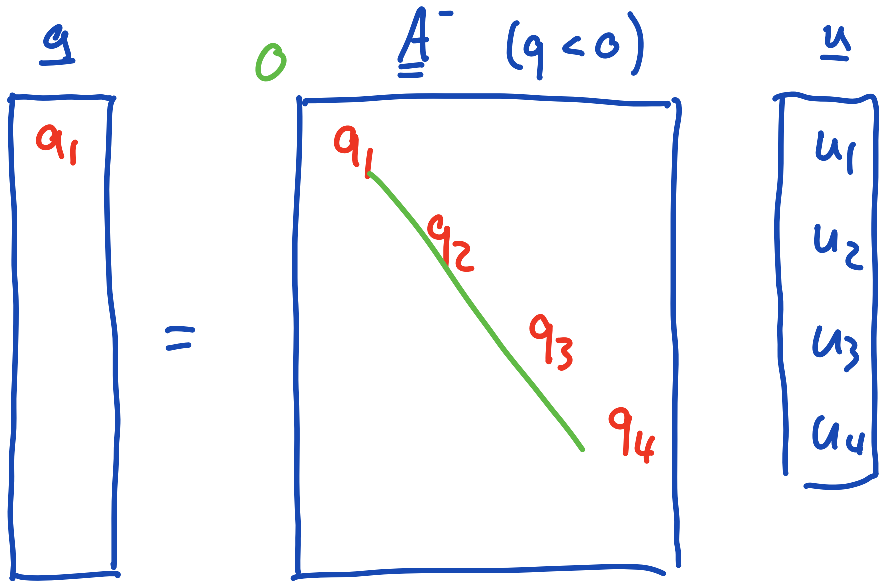
$$\underline{q} \quad\quad\quad\quad \underline{\underline{A}}^{+} \; (q>0) \quad\quad\quad\quad \underline{u}$$

$$\begin{bmatrix} \\ a_2 \\ a_3 \\ \\ \\ \end{bmatrix} = -1 \begin{bmatrix} & & & \\ q_2 & 0 & 0 & 0 \\ & q_3 & & \\ & & q_4 & \\ & & & q_5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ a_3 \\ u_4 \end{bmatrix}$$

$a_2 = q_2 \; u_1 \qquad\qquad a_2 = u_1$

$a_3 = q_3 \; u_2$

$a_1 = ?$



Need switch between $\underline{\underline{A}}^+$ and $\underline{\underline{A}}^-$ on a ~~cell ba~~ face basis depending on sign of $\underline{q}$

Build pos. & neg. flux vector

$\underline{qn} = \min(q(1:Nx), 0)$

$\underline{qp} = \max(q(2:Nx+1), 0)$

Nx+1