Lecture 20: Transient Advection-Diffusion

Logistics: - HW 9 due Th

⇒ need to post lecture notes

- Last time: Steady advection-diffusion

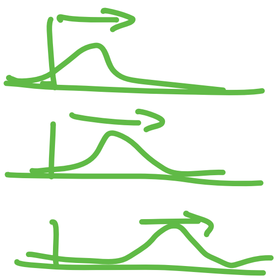
$$\nabla\cdot(\underline{v}T - \kappa\nabla T) = 0$$

MOC

→ travelling wave sol'n

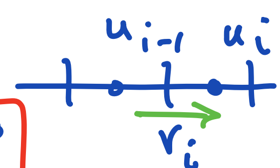
$$\underbrace{\underline{\underline{D}}\cdot\left(\underline{\underline{A}}(\underline{v}) - \underline{\underline{Kd}} * \underline{\underline{G}}\right)}\underline{u} = \underline{0}$$

$$\underline{\underline{L}}\,\underline{u} = \underline{0}$$



- A = advection matrix
- upwind flux

$$\boxed{a_i = \begin{cases} v_i u_{i-1}, & v_i \geq 0 \\ v_i u_i, & v_i \leq 0 \end{cases}}$$



- Today: Transient advection-diffusion
  - Analytic solution
  - CFL condition
  - Numerical diffusion
  - Geotherm with erosion

Transient Advection-Diffusion

Energy balance:

$$\overline{\rho c_p}\,\frac{\partial T}{\partial t} + \nabla\cdot\left[\varphi \rho_f c_{p,f}\, T - \bar{\kappa}\, \nabla T\right] = \rho H$$

ρH_0

if $\overline{\rho c_p} = \text{const.}$

$$\Rightarrow \boxed{\frac{\partial T}{\partial t} + \nabla\cdot\left[\underline{v_e}\, T - \underline{\bar{\alpha}}\, \nabla T\right] = \cancel{\rho H}\; 0}$$

$\bar{\alpha} = \dfrac{\bar{\kappa}}{\overline{\rho c_p}}$ $\qquad$ $\underline{v_e} = v_f\, \dfrac{\rho_f c_{pf}}{\overline{\rho c_p}}$

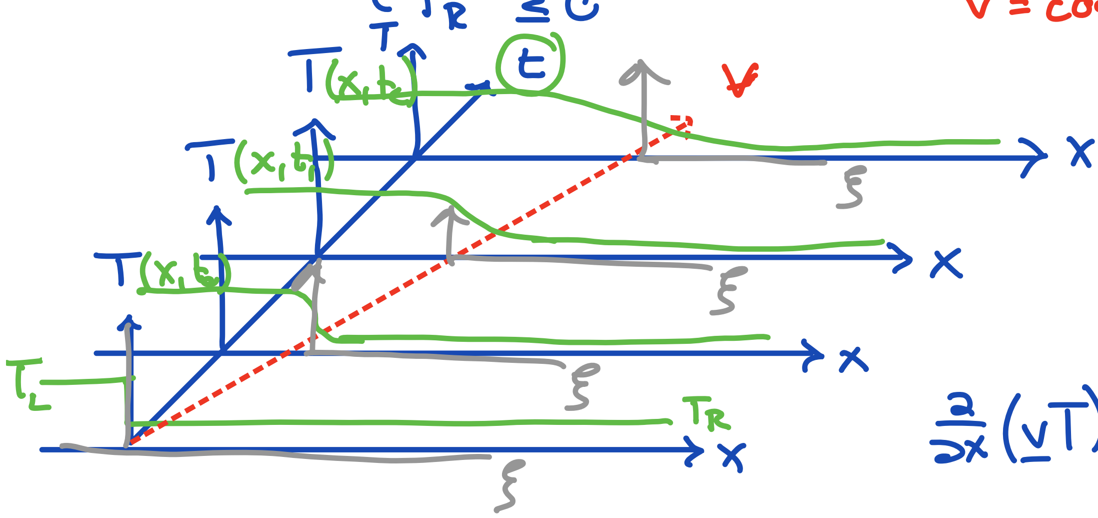
Analytic soln for Thermal front

PDE: $\dfrac{\partial T}{\partial t} + \dfrac{\partial}{\partial x}\left(v T - \underline{\alpha}\, \dfrac{\partial T}{\partial x}\right) = 0$ $\qquad -\infty < x < \infty$

IC: $T = \begin{cases} T_L & < 0 \\ T_R & \geq 0 \end{cases}$

$v \geq 0$

$v = \text{const}$



$$\frac{\partial}{\partial x}\left(\underline{v} T\right) = v\,\frac{\partial T}{\partial x} + \cancel{T\,\frac{\partial v}{\partial x}}$$

T profile is shifting to right due to advection and front is widening due to conduction.

Introduce the travelling wave coordinate

$\xi = x - vt \qquad T(x,t) = \theta(\xi(x,t),t)$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} - \alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial t}\big(\theta(\xi(x,t),t)\big) + v\frac{\partial}{\partial x}\big(\theta(\xi(x,t),t)\big) - \alpha\frac{\partial^2}{\partial x^2}\big(\theta(\xi(x,t),t)\big)$$

$$\frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial\xi}\cancelto{-v}{\frac{\partial\xi}{\partial t}} + v\frac{\partial\theta}{\partial\xi}\cancelto{1}{\frac{\partial\xi}{\partial x}} - \alpha\frac{\partial^2\theta}{\partial x^2}\cancelto{1}{\left(\frac{\partial\xi}{\partial x}\right)^2} = 0$$

$\xi = x - vt$

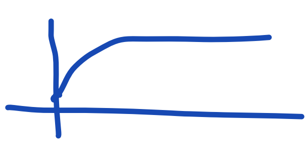
$\frac{\partial\xi}{\partial x} = 1 \qquad \frac{\partial x}{\partial t} = -v$

substitute

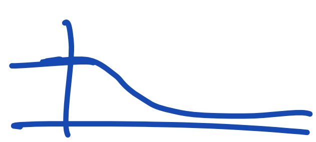
$$\frac{\partial\theta}{\partial t} - v\cancel{\frac{\partial\theta}{\partial\xi}} + v\cancel{\frac{\partial\theta}{\partial\xi}} - \alpha\frac{\partial^2\theta}{\partial\xi^2} = 0$$

$\Rightarrow \quad \boxed{\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial\xi^2}}$ Heat equation

Lecture 26



erf



$1 - \mathrm{erf} = \mathrm{erfc}$

solu: $\Theta(\xi, t) = \Theta_L + \frac{\Delta\Theta}{2} \operatorname{erfc}\left(\frac{\xi}{\sqrt{4\alpha t}}\right)$

subst: $\xi = x - vt$

$$\boxed{T(x,t) = T_L + \frac{(T_R - T_L)}{2} \operatorname{erfc}\left(\frac{x - vt}{\sqrt{4\alpha t}}\right)}$$

Numerical solu to ADE

$$\frac{\partial T}{\partial t} + \nabla \cdot \left(\underline{v}_e T - \alpha \nabla T\right) = \rho H$$

discrete ops + θ method: $\underline{u} = T(\underline{x}_c)$

$$\underline{\underline{I}}\left(\underline{u}^{n+1} - \underline{u}^n\right) + \Delta t \left(\overbrace{\underline{\underline{D}} * \underline{\underline{A}}(\underline{v}) - \underline{\underline{\mathrm{Kd}}}\ \underline{\underline{G}}}^{L}\right)\left(\theta \underline{u}^n + (1-\theta)\underline{u}^{n+1}\right) = \Delta t \underline{f}_s$$

$$\underbrace{\left(\underline{\underline{I}} + \Delta t (1-\theta) \underline{\underline{L}}(\underline{v})\right)}_{\underline{\underline{\mathrm{IM}}}(\underline{v})} \underline{u}^{n+1} = \underbrace{\left(\underline{\underline{I}} - \Delta t\, \theta\, \underline{\underline{L}}(\underline{v})\right)}_{\underline{\underline{\mathrm{EX}}}(\underline{v})} \underline{u}^n + \Delta t\, \underline{f}_s$$

$$\boxed{\underline{\underline{\mathrm{IM}}}\ \underline{u}^{n+1} = \underline{\underline{\mathrm{EX}}}\ \underline{u}^n + \Delta t\, \underline{f}_s}$$

form is same as for heat equ.

at every time step

$\Rightarrow$ solve_lbup($\underline{\underline{\mathrm{IM}}}$, $\underline{\underline{\mathrm{EX}}}*u + \Delta t\, f_s$ ....)

$\theta = 1 : \quad \underline{u}^{\theta} = \underline{u}^{n} \rightarrow$ FE explicit Δt_{max}

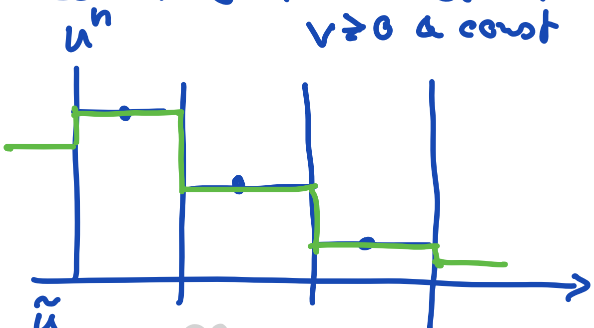
$\theta = 0 : \quad \underline{u}^{\theta} = \underline{u}^{n+1} \rightarrow$ BE implicit

$\theta = \frac{1}{2} : \quad \underline{u}^{\theta} = \frac{u^{n} + u^{n+1}}{2} \rightarrow$ CN implicit $\rightarrow$ 2nd order

Explicit time step restriction for advection

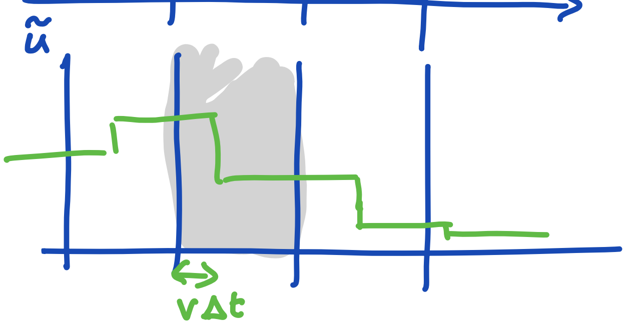
Consider the evolution of discrete solution

$v \geq 0$ & const



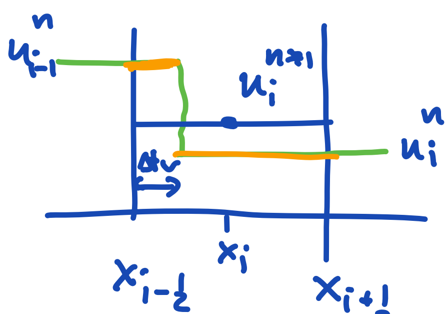
Value at cell centers represents the cell average.

$\Rightarrow$ series of steps,



Each step moves with advection velocity v

its location is $x_f + v\Delta t$
$\uparrow$
face location



New concentration at end of time step is average of profile

$$u_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{u} \, dx = \frac{1}{\Delta x} \left[\int_{x_{i-\frac{1}{2}}}^{x_{i-\frac{1}{2}} + \Delta t v} u_{i-1}^{n} \, dx + \int_{x_{i-\frac{1}{2}} + \Delta t v}^{x_{i+\frac{1}{2}}} u_i^{n} \, dx \right]$$

$$u_i^{n+1} = \frac{1}{\Delta x}\left[u_{i-1}^n \Delta t\, v + u_i^n (\Delta x - v \Delta t) \right]$$

$$\boxed{u_i^{n+1} = \frac{v\Delta t}{\Delta x} u_{i-1}^n + \left(1 - \frac{v \Delta t}{\Delta t}\right) u_i^n}$$

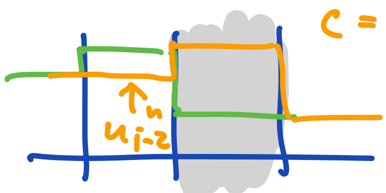
$$\frac{v\Delta t}{\Delta \cancel{x}} = \text{CFL number} = c$$

$$u_i^{n+1} = c\, u_{i-1}^n + (1-c)\, u_i^n$$

= new solution is weighted average / interpolation between u_i^n and u_{i-1}^n

if $c = \frac{v \Delta t}{\Delta x} > 1$

$c = 1$



$c \le 1$

$$\boxed{\Delta t \le \frac{\Delta x}{v}}$$