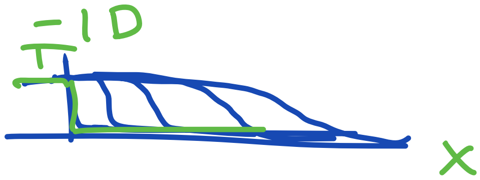
Lecture 22: Advection in 2D

Logistics:
- HW9 due (9/11)
- HW10 ⇒ transient ADE due 4/13
- (HW11 ⇒ Darcy with gravity due 4/20)
- (HW12 ⇒ convection due 4/27)

Last time:
- Transient ADE – 1D
  - Analytic soln.



    ⇒ travelling wave coord.
  - Numerical soln

$$\boxed{\begin{aligned} \underline{\underline{IM}}\,\underline{u}^{n+1} &= \underline{\underline{EX}}\,\underline{u}^{n} + \Delta t\, \underline{f}_s \\ \underline{\underline{IM}} &= \underline{\underline{S}} + \Delta t\,(1-\theta)\,\underline{\underline{L}} \\ \underline{\underline{EX}} &= \underline{\underline{S}} - \Delta t\,\theta\,\underline{\underline{L}} \end{aligned}}$$

same for any linear problem

Difference: $\boxed{\underline{\underline{L}} = \underline{\underline{D}}\left(\underline{\underline{A}}(\underline{v}) - \underline{\underline{Kd}}\,\underline{\underline{G}}\right)}$

- CFL cond.: $\Delta t \leq \frac{\Delta x}{v}$

Today: Advection in 2D

Advection matrix in 2D

Problem: In $\underline{\underline{D}}$ and $\underline{\underline{G}}$ the matrix blocks are identical, but in 2D $\underline{\underline{A}}$ matrix each block has same structure (0,& 1) but the values differ because $\underline{v}(q)$ varies across the domain.

Solution: Separate the information about block structure from magnitudes.

The overall scheme for computing the advective fluxes: $\underline{q} = \underline{\underline{A}}(\underline{v})\,\underline{u}$

$\underline{q}$

$$\begin{bmatrix} \underline{q}_x \\ \underline{q}_y \end{bmatrix} =$$

$N_f \times 1$

$$\underset{N_f \times 1}{\begin{bmatrix}\underline{a_x}\\ \underline{a_y}\end{bmatrix}} = \overbrace{\left(\underset{N_f \times N_f}{\underline{\underline{Qd}}^{+}} \cdot \underbrace{\underset{N_f \times N}{\begin{bmatrix}\underline{\underline{Ax}}^{+}\\ \underline{\underline{Ay}}^{+}\end{bmatrix}}}_{\underline{\underline{A}}^{+}} + \underset{N_f \times N_f}{\underline{\underline{Qd}}^{-}} \cdot \underbrace{\underset{N_f \times N}{\begin{bmatrix}\underline{\underline{Ax}}^{-}\\ \underline{\underline{Ay}}^{-}\end{bmatrix}}}_{\underline{\underline{A}}^{-}} \right)}^{\underline{\underline{A}}} \underset{N \times 1}{\underline{u}}$$

$$\underline{a} = \underbrace{\left(\underline{\underline{Qd}}^{+} \underline{\underline{A}}^{+} + \underline{\underline{Qd}}^{-} \underline{\underline{A}}^{-} \right)}_{\underline{\underline{A}} = \underline{\underline{Qd}}^{+} \underline{\underline{A}}^{+} + \underline{\underline{Qd}}^{-} \underline{\underline{A}}^{-}} \underline{u}$$

Where we have following sparse matrices:

$\underline{\underline{Qdp}}$ = $Nf \times Nf$ matrix with the pos. fluxes on the diag.

$\underline{\underline{Qdn}}$ = $Nf \times Nf$ matrix with the neg. fluxes on the diag

$\underline{\underline{Ap}}$ - $Nf \times N$ matrix with ones in locations of pos. fluxes

$\underline{\underline{An}}$ = $Nf \times N$ matrix with ones in locations of neg. flux.

If flow is evolving only $\underline{\underline{Qdp}}$ and $\underline{\underline{Qdn}}$ have to be updated, but $\underline{\underline{Ap}}$ & $\underline{\underline{An}}$ remain the same.

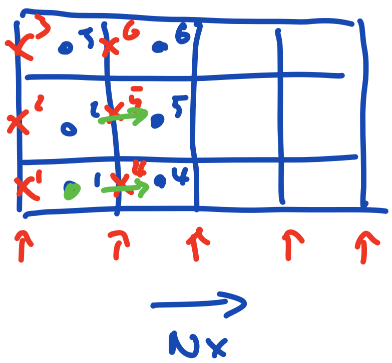
$$\underline{\underline{Qdp}} = \text{spdiags}(\max(q,0), 0, Nf, Nf)$$
$$\underline{\underline{Qdn}} = \text{spdiags}(\min(q,0), 0, Nf, Nf)$$

$$\boxed{\underline{\underline{A}}(q) = \underline{\underline{Qdp}}(q)\,\underline{\underline{Ap}} + \underline{\underline{Qdn}}(q)\,\underline{\underline{An}}}$$

$$\underline{\underline{Ap}} = \begin{bmatrix} \underline{\underline{Axp}} \\ \underline{\underline{Ayp}} \end{bmatrix} \qquad \underline{\underline{An}} = \begin{bmatrix} \underline{\underline{Axn}} \\ \underline{\underline{Ayn}} \end{bmatrix}$$

The 4 blocks $\underline{\underline{Axp}}$, $\underline{\underline{Axn}}$, $\underline{\underline{Ayp}}$, $\underline{\underline{Ayn}}$ can be built by kronecker/tensor products.

Ax matrices

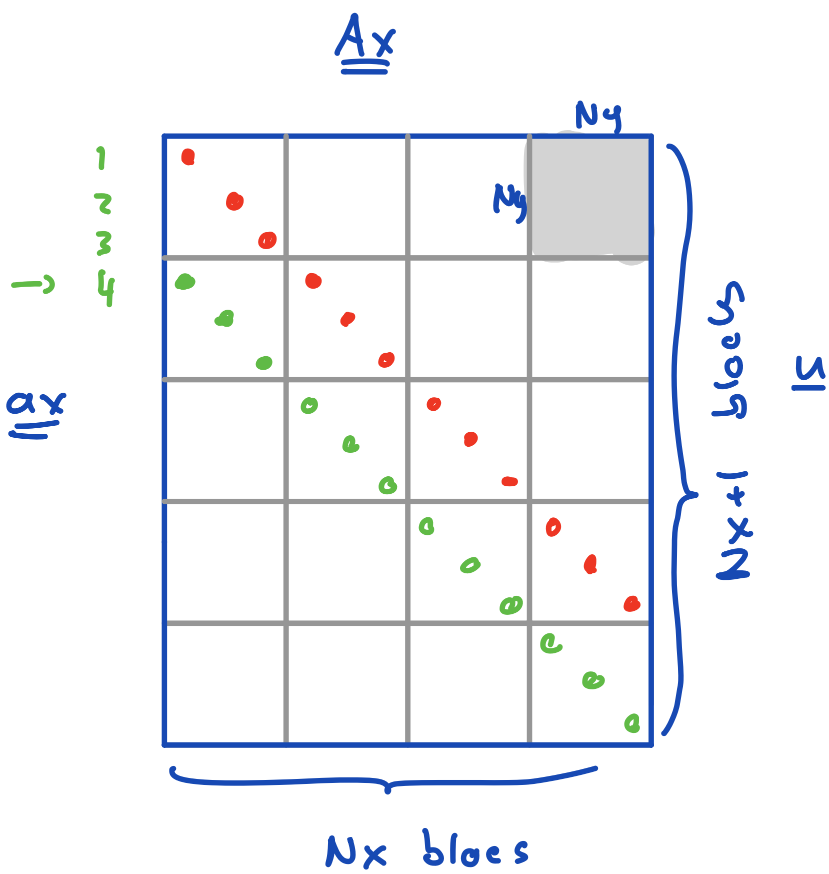


Ax computes Ny by (Nx+1) fluxes from Nx·Ny temp.

Nx columns of Ny temps

Nx+1 columns of Ny flux

• neg fluxes



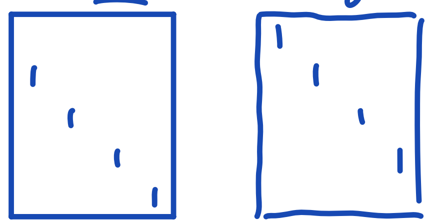
Axp 1's where green dots are

Axm 1's where red dots are

$\Rightarrow \quad \underline{\underline{I_y}} = \text{speye}(N_y)$

In 1D:

$\underline{\underline{A_{xp1}}}$ $\quad\quad$ $\underline{\underline{A_{xn1}}}$

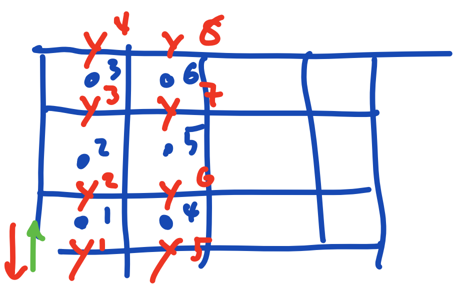


2D matrices: $\underline{\underline{A_{xp}}} = \text{kron}(\underline{\underline{A_{xp1}}}, \underline{\underline{I_y}})$

$\underline{\underline{A_{xn}}} = \text{kron}(\underline{\underline{A_{xn1}}}, \underline{\underline{I_y}})$

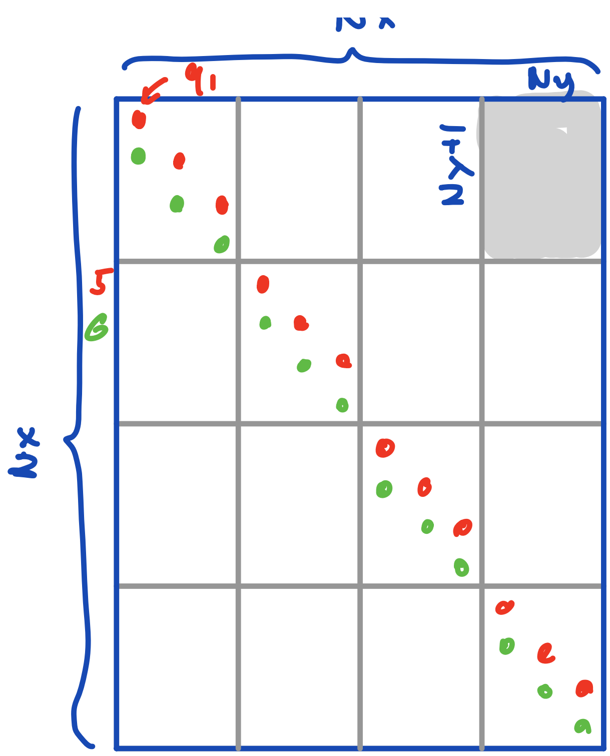
$\underline{\underline{A_y}}$ matrices

A_y computes N_x columns of N_y+1 fluxes from N_x columns of N_y temps



$\Rightarrow$ $\underline{\underline{A_y}}$ is N_x by N_x block matrix with blocks of size N_y+1 by N_y

$\underline{a_y} = \underset{N_x}{\underline{\underline{A_y}}} \; \underline{u}$



• neg. fluxes

• pos. fluxes

Overall block structure

$\underline{\underline{I_x}} = \text{speye}(N_x)$

Each block is 1D $\underline{A}$ mat.

Ayp1 = spdiags(ones(Ny,1),-1,Ny+1,Ny)

Ayn1 = spdiags(ones(Ny,1),0,Ny+1,Ny)

Assemble 2D matrices:

$\underline{\underline{A_{yp}}} = \text{kron}(\underline{\underline{I_x}}, \underline{\underline{A_{yp1}}})$

$\underline{\underline{A_{yn}}} = \text{kron}(\underline{\underline{I_x}}, \underline{\underline{A_{yn1}}})$

A

Assemble overall 2D pos. & neg. matrices:

$$\underline{\underline{A_p}} = \begin{bmatrix} \underline{\underline{A_{xp}}} \\ \underline{\underline{A_{yp}}} \end{bmatrix} \qquad \underline{A_n} = \begin{bmatrix} \underline{\underline{A_{xn}}} \\ \underline{\underline{A_{yn}}} \end{bmatrix}$$

$$\boxed{\underline{A}(q) = \underline{\underline{Q_{dp}}}(q) * \underline{\underline{A_p}} + \underline{\underline{Q_{dn}}}(q) * \underline{\underline{A_n}}}$$

$\Rightarrow$ flux_upwind