Lecture 23: Convection in Porous media

Logistics: – HW 10 only one question
still working on second one
⇒ next home work

Last time: – 2D advection operator

$$\underline{a} = \underline{\underline{A}}(\underline{v})\,\underline{u}$$

– separate location of entry from value

$$\boxed{\begin{aligned} &\underline{\underline{A}} = \underline{\underline{Q}}_p(\underline{v})\,\underline{\underline{A}}_p + \underline{\underline{Q}}_n(\underline{q})\,\underline{\underline{A}}_n \\ &\underline{\underline{A}}_p = \begin{bmatrix} \underline{\underline{A}}_{xp} \\ \underline{\underline{A}}_{yp} \end{bmatrix} \qquad \underline{\underline{A}}_n = \begin{bmatrix} \underline{\underline{A}}_{xn} \\ \underline{\underline{A}}_{yn} \end{bmatrix} \end{aligned}}$$

kronecker prod: $\underline{\underline{A}}_{xp}$ $\underline{\underline{A}}_{xn}$ $\underline{\underline{A}}_{yp}$ $\underline{\underline{A}}_{yn}$

Today: Thermally-driven convection
in porous media

Convection in porous media

Lectures: 4 - 14 Incompressible groundwater flow (mass balance)

$$-\nabla \cdot (K \nabla h) = f_s$$

$$\underline{q} = -K \nabla h$$

$\Rightarrow$ solve for $h \rightarrow \underline{q}$ (Flow problem)

Lectures: 15 - 22 Energy balance

$$\overline{\rho c_p} \frac{\partial T}{\partial t} + \nabla \cdot [\underline{q} \rho_f c_{p,f} T - \bar{\kappa} \nabla T] = 0$$

$\Rightarrow$ solve for T (Transport problem)

Flow always influences transport through $\underline{q}$.

Convection transport has feedback on the flow.

$\Rightarrow$ non-linear problem

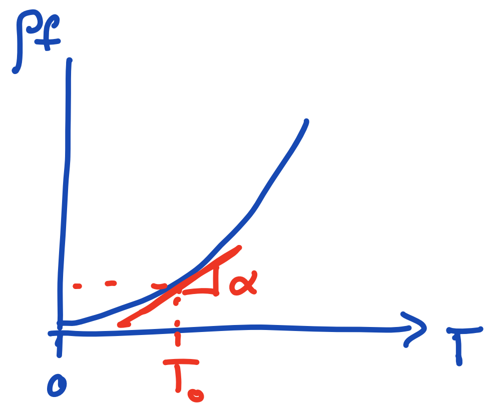
Convective flow is induced by density variations due to temperature changes.

Equation of state for $\rho_f = \rho_f(T)$, here we consider simple linear change

$$\boxed{\rho_f(T) = \rho_0 (1 + \alpha T)} \qquad \rho_0 \left(1 + \alpha_{T_0} (T - T_0)\right)$$

New physical property: thermal expansivity

$$\alpha = \frac{1}{V} \frac{\partial V}{\partial T}\Big|_p \qquad \left[\frac{1}{K}\right]$$



for water: $\alpha \approx 1.5 \cdot 10^{-4} \frac{1}{K}$

$\Delta T \sim 10^{2} \rightarrow \Delta\rho \approx 10^{-2} \frac{kg}{m^3}$

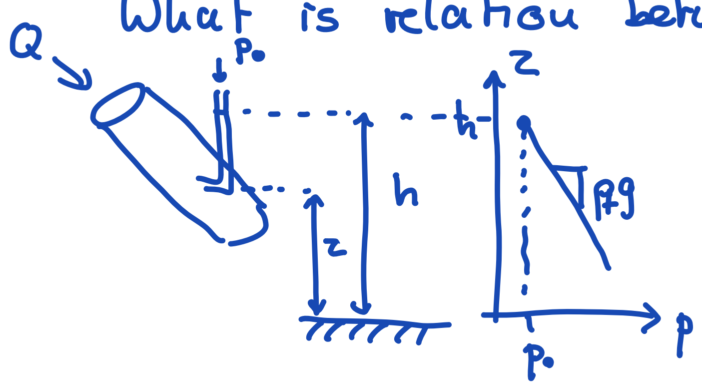
How does this enter the flow problem

mass: $\nabla \cdot \underline{q} = f_s$

Darcy: $\underline{q} = -K \nabla h$

⇒ write Darcy's law in terms of pressure

What is relation between h and p?



hydrostatic pressure:

$$p(z) = p_0 + \rho_f g (h - z)$$

solve for head: $\boxed{h = \frac{p - p_0}{\rho_f g} + z}$

Substitute into Darcy's law:

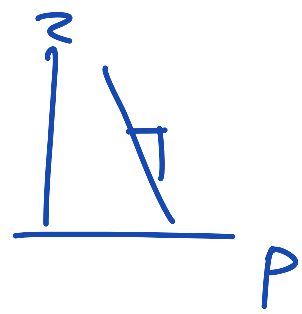
$$q = -K \nabla \left(\frac{p - p_0}{\rho_f g} + z \right)$$

$$q = -\underbrace{\frac{K}{\rho_f g}}_{\frac{k}{\mu_f}} \nabla \left(p - p_0 + \rho_f g z \right)$$

k = intrinsic permeability of rock

μ_f = viscosity of fluid

$$q = -\frac{k}{\mu_f} \left(\nabla p - \cancelto{0}{\nabla p_0} + \rho_f g \cancelto{\hat{z}}{\nabla z} \right)$$



Darcy's law in pressure form

$$\boxed{q = -\frac{k}{\mu_f} \left(\nabla p + \rho_f g \hat{z} \right)}$$

$$q = -K \nabla h$$

Note: $h = \text{const} \Rightarrow q = 0$

$q \parallel -\nabla h$

$\Big\}$ h is flow potential

$p = \text{const.} \longrightarrow q \neq 0$ $\quad q = -\frac{k}{\mu} \rho_f g \hat{z}$

$q = 0$ if p = hydrostatic $\quad \nabla p = \rho_f g \hat{z}$

Substitute into mass balance:

$$\nabla \cdot q = 0$$

$$-\nabla \cdot \left[\frac{k}{\mu_f} \left(\nabla p + \rho_f g \hat{z} \right) \right] = f_s \qquad \lambda = \frac{k}{\mu_f} \text{ mobility}$$

$\rho_f g \hat{z}$ is known $\rightarrow$ rhs

$$\boxed{-\nabla \cdot \left[\frac{k}{\mu_f} \nabla p \right] = f_s + \underbrace{\nabla \cdot \left[\rho_f g \hat{z} \right]}_{\text{buoyancy term}}}$$

$$\nabla \cdot \left[\rho_f g \hat{z} \right] = g \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{pmatrix} 0 \\ 0 \\ \rho_f \end{pmatrix} = g \frac{\partial \rho_f}{\partial z}$$

Coupled non-linear system of equations

mass bal.: $-\nabla\cdot\left[\frac{k}{\mu_f}\nabla p\right] = f_s + \nabla\cdot\left[\rho_f(T)\, g\, \hat{z}\right]$

energy bal.: $\overline{\rho c_p}\,\frac{\partial T}{\partial t} + \nabla\cdot\left[q(p)\,\rho_f c_{p,f}\, T - \bar{\kappa}\,\nabla T\right] = 0$

const. equs.:

1) Darcy: $q = -\frac{k}{\mu_f}\left(\nabla p + \rho_f g\, \hat{z}\right)$

2) EoS: $\rho_f = \rho_0\left(1 - \alpha\,(T - T_0)\right)$

Boussinesq approximation:

Because the density changes are small we neglect them everywhere except in buoyancy term.

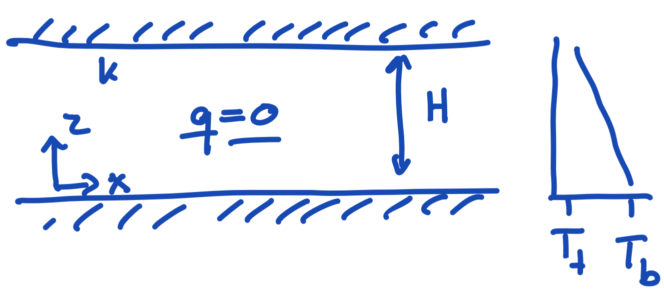
$\rho_f = \rho_0$

$\Rightarrow \overline{\rho c_p} = \phi\,\rho_0 c_{p,f} + (1-\phi)\,\rho_r c_{p,r}$

A RhoCp

$$\underline{\overline{\rho c_p}}\,\frac{\partial T}{\partial t} + \nabla\cdot\left[q\,\rho_0\, c_{p,f}\, T - \bar{\kappa}\,\nabla T\right] = 0$$

Canonical problem:



assume $L \gg H$

$$\boxed{T(\underline{x},0) = T_b - (T_b - T_t)\,\frac{z}{H}}$$

Initial condition

Boundary conditions: $T(z=0,t) = T_b$

$T(z=H,t) = T_t$

$\underline{q} \cdot \underline{n} = \underline{0}$

$\Rightarrow$ Next time solve pressure based flow problem.