Lecture 3: Introduction to numerics

Logistics: - office hrs: Mon 10-11am (in person)
Wed 3-4pm (on zoom)
- docs: you should have been added automatically

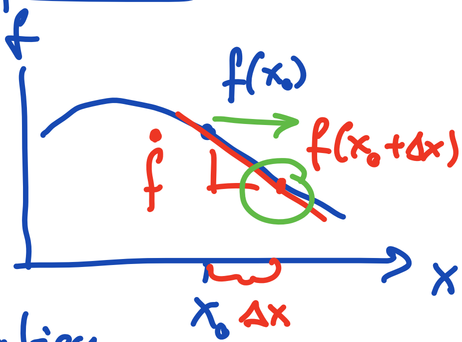
Last time: - Darcy's law: $\boxed{\underset{\sim}{q} = -K\nabla h}$
- General balance law: $\frac{\partial u}{\partial t} + \nabla\cdot \underset{\sim}{j} = \hat{f}$
- pore fluid mass balance: $\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot\underset{\sim}{q} = \hat{f}$
- incompressible flow: $\boxed{-\nabla\cdot K\nabla h = f}$ Poisson Laplace

Today: - Intro finite differences
- Differentiation matrices
- Example: Flow near injection well
$\Rightarrow$ Conservative Finite Differences

Aim: Motivate approach

Introduction to finite differences

In calculus we define



$$\dot{f}(x_0) = \left.\frac{df}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$\Rightarrow$ infinitesimal

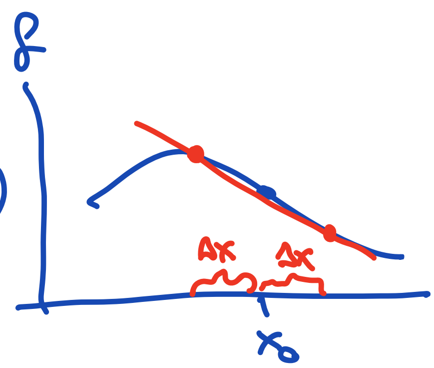
In finite difference approximation

$$\hat{\dot{f}}(x_0) = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \underbrace{O(\Delta x)}_{\text{error}}$$

In proper num. methods class you prove that this "one-sided" approx. is first-order accurate

$\Rightarrow$ error decreasing as $\frac{1}{\Delta x}$

Central finite difference



$$\hat{\dot{f}}(x_0) = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

$\Rightarrow$ second order accurate

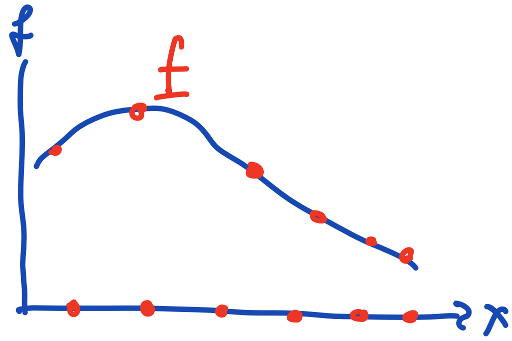
error $\sim \frac{1}{\Delta x^2}$

$\Rightarrow$ go to approximation

Differentiation Matrix

The derivative is linear differential operator

$$\underbrace{\dot{f}(x)}_{\text{function}} = \underbrace{\mathcal{D}}_{\text{operator}}[\underbrace{f(x)}_{\text{function}}]$$



The discrete equivalent of function f. is the vector $\underline{f} = f(\underline{x})$. Similarly we can define $\underline{df} = \dot{f}(\underline{x})$

What is the discrete equivalent of $\mathcal{D}$?

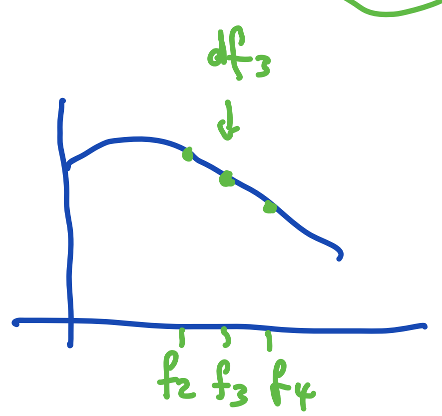
$$\underline{df} = \boxed{\underline{\underline{D}}}\ \underline{f}$$

has to be a matrix, because it is linear and relates two vectors to each other

$$df_3 = \frac{f_4 - f_2}{2\Delta x}$$

⇒ Differentiation matrix $\underline{\underline{D}}$

$$\underline{df} \begin{bmatrix} \vdots \\ df_3 \\ \vdots \end{bmatrix} = \underline{\underline{\frac{1}{2\Delta x}}} \overset{\underline{\underline{D}}}{\begin{bmatrix} \vdots & & & & & \\ -1 & & 1 & & & \\ & -1 & & 1 & & \\ & & -1 & & 1 & \\ & & & -1 & & 1 \\ & & & & & \ddots \end{bmatrix}} \begin{bmatrix} f_2 \\ f_3 \\ f_4 \end{bmatrix} \underline{f}$$



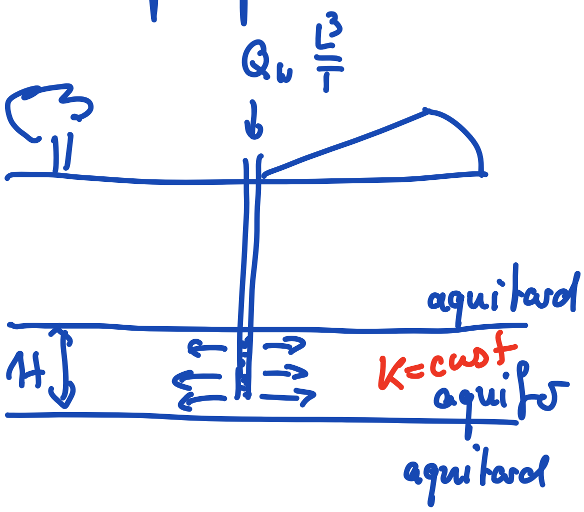
⇒ $\underline{\underline{D}}$ has very simple bi-diagonal structure

Note: bound. need extra work.

What about 2nd derivative? $\ddot{f} = \frac{d^2 f}{dx^2}$

$$\underline{ddf} = \underline{\underline{D}}\,\underline{df} = \underline{\underline{D}}\,\underline{\underline{D}}\,\underline{f} = \underline{\underline{D}}^2\,\underline{f}$$

Example problem: Flow around an injection well



cylindrical geometry

$$-\nabla\cdot(K\nabla h) = f$$

PDE: $-\frac{1}{r}\frac{d}{dr}\left(rK\frac{dh}{dr}\right) = 0$

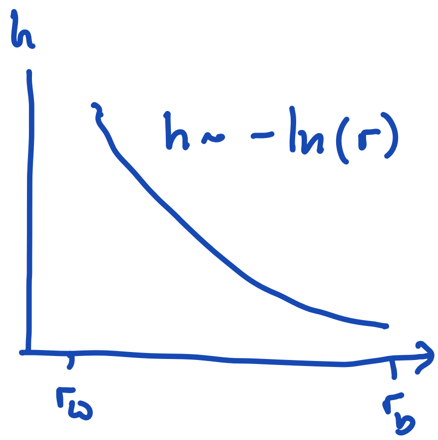
$r \in [r_w, r_b]$

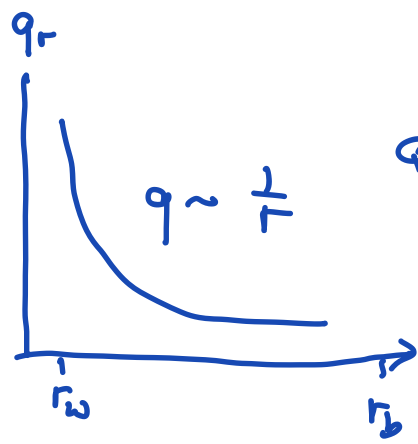
BC: $Q_w = A_w\, q(r_w)$

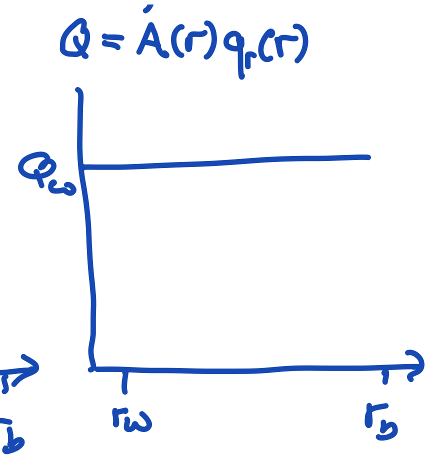
⇒ $\left.\frac{dh}{dr}\right|_{r_w} = -\frac{Q_w}{A_w K}$ $\quad A_w = 2\pi r_w H$

$h(r_b) = h_b$

Injection ~~into~~ with const. rate Q_w into a well with radius r_w.







Finite difference discretization

$$-\cancel{\frac{1}{r}}\frac{d}{dr}\left(r\cancel{k}\frac{dh}{dr}\right)=0$$

$$\frac{d}{dr}\left(r\frac{dh}{dr}\right)=0$$

$$\frac{dh}{dr}+r\frac{d^2h}{dr^2}=0$$

Discretize: $\underline{\underline{D}}\,\underline{h}+\underline{\underline{R}}\;\underline{\underline{D}}^2\,\underline{h}=0$

$$\begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & r_3 & \\ & & & \ddots \end{pmatrix}$$

$$(\underbrace{\underline{\underline{D}}+\underline{\underline{R}}\,\underline{\underline{D}}^2}_{\underline{\underline{L}}})\,\underline{h}=0 \qquad \underline{\underline{L}}\,\underline{h}=0$$

Need to impose $\left.\frac{dh}{dr}\right|_{r_w}$ note need to use "one sided derivative"