Lecture 5: Shallow aquifer model

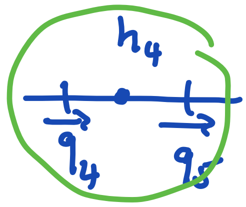
Logistics: - HW1 due Thursday   P1: 4/13

P2: 0/13

- Thanks for Piazza discussion!
- Late policy is $\boxed{10\% \text{ off}}$

Last time: - Discrete operators

- Staggered grid



$\Rightarrow$ second order

- Discrete 1D Divergence & Gradient

Check how to save work in matlabgrader

$$\underline{\underline{D}} = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix} \qquad \underline{\underline{G}} = \frac{1}{\Delta x}\begin{bmatrix} 0 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & 0 \end{bmatrix}$$

$$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T} \text{ in interior}$$

Today: - Mean operator $\underline{\underline{M}}$

- Introduce "shallow aquifer model"

$\Rightarrow$ good first example problem to solve numerically

- Boundary conditions

Mean operator

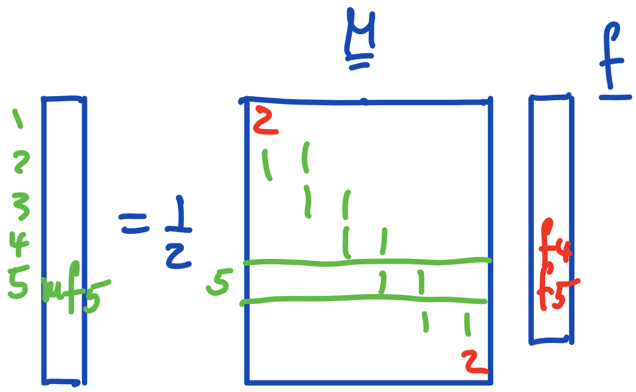
⇒ variable coefficients $\quad K = K(\underline{x})$

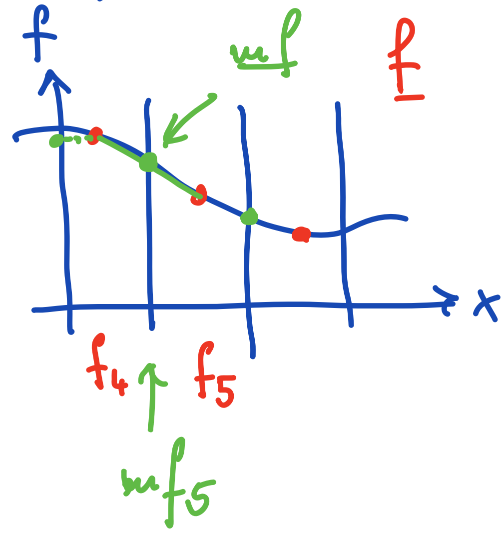
$$\underline{mf} = \underline{\underline{M}}\,\underline{f}$$

cell faces

same shape as $\underline{\underline{G}}$

cell centers

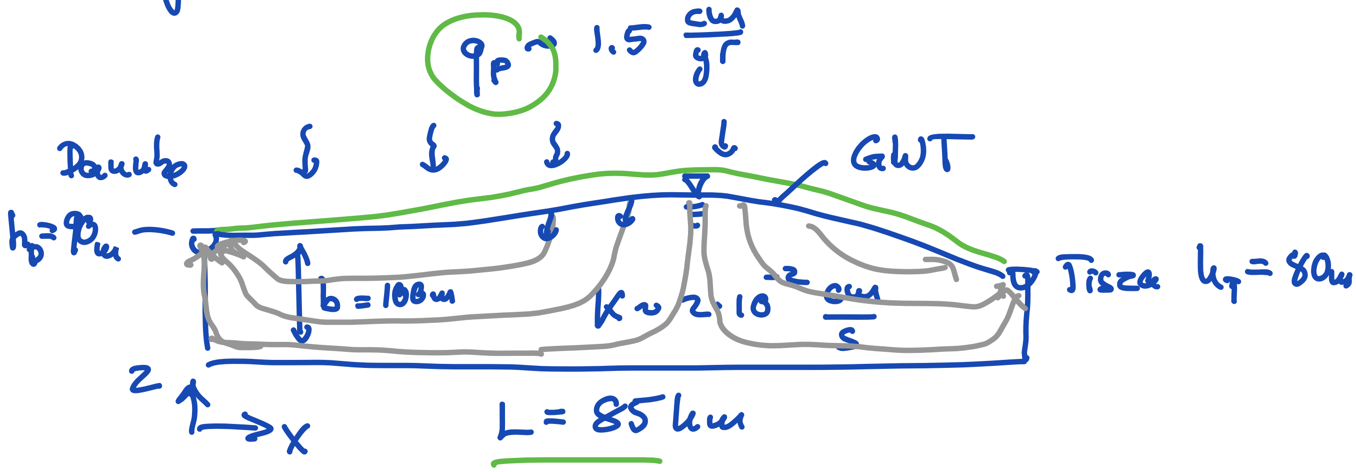




$$mf_5 = \frac{f_4 + f_5}{2}$$

Groundwater recharge between two rivers

(Sanford et al 2001)



Aquifer aspect ratio: $b/L = \frac{100}{85{,}000} = \frac{1}{850} \sim 0.001$

$\Rightarrow$ flow is mostly horizontal $\Rightarrow$ 1D model

Scaling analysis

$x_D = \frac{x}{L}$ $\quad z_D = \frac{z}{b}$ $\quad q_{x,D} = \frac{q_x}{q_{x,c}}$ $\quad q_{zD} = \frac{q_z}{q_{z,c}}$

↑ characteristic

$$\nabla \cdot q = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0$$

$$= \frac{\partial (q_{x,c}\, q_{x,D})}{\partial (L\, x_D)} + \frac{\partial (q_{z,c}\, q_{zD})}{\partial (b\, z_D)} =$$

$$= \frac{q_{x,c}}{L} \frac{\partial q_{xD}}{\partial x_D} + \frac{q_{z,c}}{b} \frac{\partial q_{z,D}}{\partial z_D} = 0$$

$$\frac{\partial q_{x,D}}{\partial x_D} + \underbrace{\frac{q_{z,c}}{q_{x,c}}\,\frac{L}{b}}_{\Pi = 1}\,\frac{\partial q_{z,D}}{\partial z_D} = 0$$

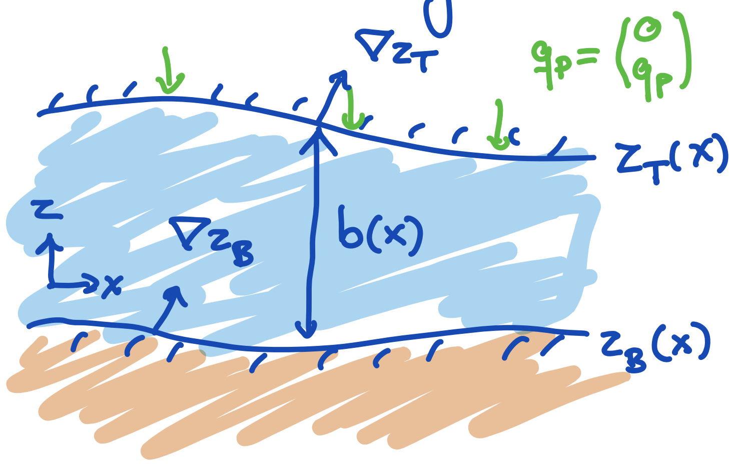
$$\Rightarrow \boxed{\underline{q_{z,c}} = \underset{\underset{10^{-3}}{\uparrow}}{\frac{b}{L}}\, q_{x,c}}$$

$\Rightarrow$ vertical flow can be neglected (on average)

Assume: $\quad q_z = 0 \quad \underset{\underset{q_z = -K\frac{\partial h}{\partial z}}{\uparrow}}{\Rightarrow} \quad \frac{\partial h}{\partial z} = 0 \quad \Rightarrow \quad h = h(x)$

Darcy: $\quad q_x = -K\,\frac{\partial h}{\partial x}$

Vertical integration:



$$b(x) = z_T(x) - z_B(x)$$

$$\int_{z_B(x)}^{z_T(x)} \nabla \cdot q \, dz \overset{?}{=}$$

$\Rightarrow$ Leibnitz integral rule

$$\int_{z_B}^{z_T} \nabla\cdot q = \underbrace{\nabla \cdot \int_{z_B}^{z_T} q \, dz}_{\frac{\partial}{\partial x}(b\,q_x)} + \cancel{\left(q \cdot \nabla z_B\right|_{z_B}}^{\,0} - \underbrace{\left(q\cdot \nabla z_T\right|_{z_T}}_{-q_P}$$

$\begin{pmatrix} q_x(x) \\ 0 \end{pmatrix}$

$b =$

assume: 1) $\underline{q} = \begin{pmatrix} q_x \\ 0 \end{pmatrix}$ $\quad \underline{q} \neq \underline{q}(z) \quad q_x = q_x(x)$

2) bottom of aquifer is impermeable

$$\underline{q} \cdot \nabla z_B \big|_{z_B} = 0$$

3) slope of top of aquifer is small

$$\underline{q} \cdot \nabla z_T \big|_T \approx -q_p \quad \text{precip}$$

Substitute

$$\int_{z_B}^{z_T} \nabla \cdot q \, dz = \frac{\partial}{\partial x}(b q_x) + q_p = 0$$

Substitute Darcy $\quad q_x = -K \frac{\partial h}{\partial x}$

$$-\frac{\partial}{\partial x}\left(bK \frac{\partial h}{\partial x}\right) + q_p = 0$$

$$\boxed{-\frac{\partial}{\partial x}\left(\underline{\underline{b}}K \frac{\partial h}{\partial x}\right) = +\underline{\underline{q_p}}}$$

(Note has to be plus)

$\Rightarrow$ 1D model for aquifer recharge

$$\int_{z_B}^{z_T} \begin{pmatrix} q_x(x) \\ 0 \end{pmatrix} dz = \begin{pmatrix} q_x(x) \\ 0 \end{pmatrix} \underbrace{\int_{z_B}^{z_T} dz}_{} =$$

Note: • in 2D model q_p is a boundary flux

• but in 1D it becomes a rhs source term.

$bK = T$ transmissivity of aquifer

• Really we have unconfined aquifer

$z_T - z_B = b = h$

$$-\frac{\partial}{\partial x}\left(K\, h \frac{\partial h}{\partial x}\right) = q_p \qquad \text{non-linear}$$

$\Rightarrow$ here we will assume a confined aquifer $b \neq b(x) = \text{const} \Rightarrow$ linear

Simplified example problem:

$$\boxed{\begin{array}{lll} \text{PDE:} & -\dfrac{d}{dx}\left(bK \dfrac{dh}{dx}\right) = q_p & x \in [0, L] \\ \text{BC:} & h(0) = h_D & h(L) = h_T \end{array}}$$

Integrate twice:

$$h = h_D + \left(\frac{h_T - h_D}{L} + \cancelto{0}{\frac{q_p L}{2bK}}\right) x - \cancelto{0}{\frac{q_p}{2bK}}\, x^2$$

$$q = \frac{q_p}{b}\left(x - \frac{L}{2}\right) - \frac{K}{L}\left(h_T - h_D\right)$$

Sketch of solution