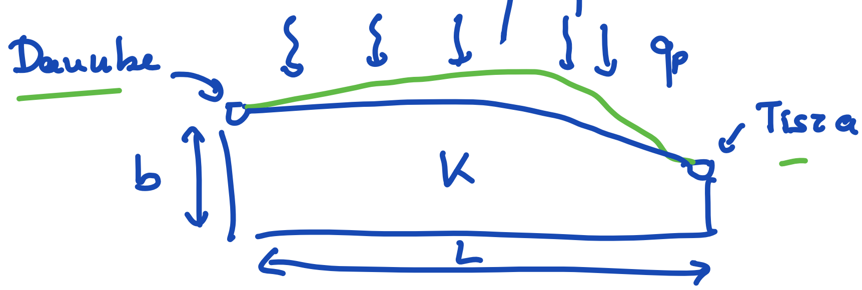
Lecture 6: Dirichlet Boundary Conditions

Logistics: - HW 1 due 11/12

if you have problems come to office hrs!

- HW 2 will be posted tonight

Last time: - Shallow aquifer model



PDE: $-\frac{d}{dx}\left(bK\frac{dh}{dx}\right) = q_p \qquad x \in [0, L]$

→ BC: $h(0) = h_D \qquad h(L) = h_T$

Dirichlet

Today: - Implementation of BC's

- Dirichlet BC set unknown on bnd

⇒ constraints

- Eliminate constraints

- Solve a reduced system

- homogeneous BC: $h(0) = h(L) = 0$

- heterogeneous BC: $h(0) = h_D \quad h(L) = h_T$

Dirichlet BC's & constraints

Homogeneous BC:

PDE: $-\frac{d}{dx}\left(bK\frac{dh}{dx}\right) = q_p \qquad x \in [0, L]$

BC: $h(0) = h(L) = 0$

Discretization of PDE:

$$-\underline{\underline{D}} \cdot bK \cdot \underline{\underline{G}}\, \underline{h} = \underline{f_s} \qquad \underline{f_s} = q_p * \mathrm{ones}(N_x,1)$$

$$\underbrace{-bK\, \underline{\underline{D}}\,\underline{\underline{G}}}\, \underline{h} = \underline{f_s}$$

$$\boxed{\underline{\underline{L}}\, \underline{h} = \underline{f_s}} \qquad \underline{\underline{L}} = -bK\, \underline{\underline{D}}\, \underline{\underline{G}}$$

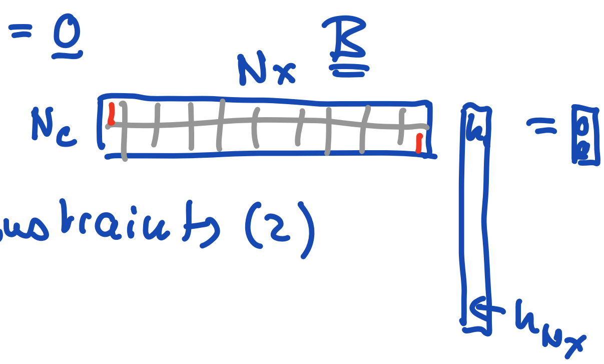
$\uparrow$ system matrix

Matlab note: $\underline{h} = \underline{\underline{L}} \backslash \underline{f_s}$

Need to write BC as linear system

$$\underline{\underline{B}}\, \underline{h} = \underline{0}$$

$h_1 = 0 \qquad h_{N_x} = 0$



N_c = number of constraints (2)

Full discrete problem:

PDE: $\boxed{\underline{\underline{L}}\,\underline{h} = \underline{f}_s}$ $\qquad \underline{\underline{L}}$ = system matrix $N_x \cdot N_x$

BC: $\boxed{\underline{\underline{B}}\,\underline{h} = \underline{0}}$ $\qquad \underline{\underline{B}}$ = constraint matrix $N_c \cdot N_x$

Neither system has a unique solution but together they do ⇒ combine them be eliminating the constraints in $\underline{\underline{B}}$ from $\underline{\underline{L}}$

Reduced Linear System

Constraints reduce the # unknowns

⇒ solve smaller system

Reduced system: $\boxed{\underline{\underline{L}}_r\,\underline{h}_r = \underline{f}_{s,r}}$

$\underline{h}_r$ is $(N_x - N_c)\cdot 1$ red. solution vector

$\underline{f}_{s,r}$ is $(N_x - N_c)\cdot 1$ red. rhs vector

$\underline{\underline{L}}_r$ is $(N_x - N_c)\cdot(N_x - N_c)$ red. system matrix

What is relation between? $\underline{h}_r$ and $\underline{h}$

$\underline{f}_{s,r}$ and $\underline{f}_s$

$\underline{\underline{L}}_r$ and $\underline{\underline{L}}$

Projection matrix

Two vectors of different length are related by rectangular matrix

$$\underset{N_x \cdot 1}{\underline{h}} = \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \quad \underset{(N_x - N_c) \cdot 1}{\underline{h}_r}$$

What is $\underline{\underline{N}}$?

For now just require that $\underline{\underline{N}}$ is orthonormal.

$$\underline{\underline{N}} = \begin{bmatrix} | & | & & | \\ n_1 & n_2 & \dots & n_i \\ | & | & & | \end{bmatrix}$$

n_i columns of $\underline{\underline{N}}$

$\underline{n}_i \cdot n_{j \neq i} = 0$

$\underline{n}_i \cdot \underline{n}_i = 1$

It follows:

a) $$\underset{(N_x - N_c) \cdot N_x}{\underline{\underline{N}}^T} \quad \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} = \underset{(N_x - N_c) \cdot (N_x - N_c)}{\underline{\underline{I}}_r}$$

b) $$\underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \quad \underset{(N_x - N_c) \cdot N_x}{\underline{\underline{N}}^T} = \underset{N_x \cdot N_x}{\underline{\underline{I}}'}$$

$\underline{\underline{I}}'$ "identity" in full space but with N_c zeros on diagonal if $\quad \underline{h} = \underline{\underline{N}}\, \underline{h}_r$

$$\underline{\underline{N}}^T \underline{h} = \underline{\underline{N}}^T \underline{\underline{N}}\, \underline{h}_r = \underline{\underline{I}}_r \underline{h}_r = \underline{h}_r$$

$\Rightarrow \quad \boxed{\begin{array}{l} \underline{h} = \underline{\underline{N}}\, \underline{h}_r \\ \underline{h}_r = \underline{\underline{N}}^T \underline{h} \end{array}}$ $\quad \underline{\underline{N}}$ allows us to go between full and reduced space/vector..

we say $\underline{\underline{N}}^T$ projects the $\underline{h}$ into the reduced solution space.

Similarly: $\quad \underline{f}_s = \underline{\underline{N}}\, \underline{f}_{s,r} \qquad \underline{f}_{r,s} = \underline{\underline{N}}^T \underline{f}_s$

How is $\underline{\underline{L}}$ projected into reduced space?

$$\underline{\underline{L}}\, \underline{h} = \underline{f}_s$$

$$\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{h} = \underline{\underline{N}}^T \underline{f}_s = \underline{f}_{s,r}$$

$$\underline{\underline{N}}^T \underline{\underline{L}}\, \underbrace{\underline{\underline{N}}\,\underline{\underline{N}}^T}_{I'} \underline{h} = \underline{f}_{s,r}$$

$$\underbrace{\underline{\underline{N}}^T \underline{\underline{L}}\, \underline{\underline{N}}}_{\underline{\underline{L}}_r} \underline{h}_r = \underline{f}_{s,r} \qquad \boxed{\underline{\underline{L}}_r = \underline{\underline{N}}^T \underline{\underline{L}}\, \underline{\underline{N}}}$$

Reduced linear system:

$$\underline{\underline{L}}_r \, \underline{h}_r = \underline{f}_{s,r}$$

$$\underline{\underline{L}}_r = \underline{\underline{N}}^T \underline{\underline{L}} \, \underline{\underline{N}}$$

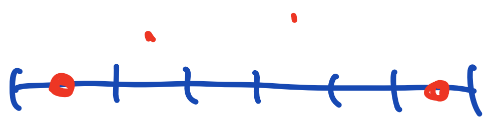
$$\underline{h}_r = \underline{\underline{N}}^T \underline{h}$$

$$\underline{f}_{s,r} = \underline{\underline{N}}^T \underline{f}_s$$

Now we just need to find $\underline{\underline{N}}$!

$\underline{\underline{N}}$ needs to contain information about the boundary conditions ($\underline{\underline{B}}$),

$$\underline{\underline{B}} \, \underline{h} = \underline{0}$$



We need solution that are in the null space of $\underline{\underline{B}}$, i.e. all solutions that satisfy $\underline{\underline{B}} \, \underline{h} = \underline{0}$

$\underline{\underline{N}}$ need to project $\underline{h}$ into the null space of $\underline{\underline{B}}$, because an $\underline{h}$ not in $\mathcal{N}(\underline{\underline{B}})$ does not satisfy BC's!

The matrix $\underline{\underline{N}}$ can be <u>any</u> orthonormal basis for null space of $\underline{\underline{B}}$.

In Matlab we can find nullspace

$$\underline{\underline{N}} = \text{null}(\underline{\underline{B}})$$

$$\underline{\underline{N}} = \text{spnull}(\underline{\underline{B}}) \quad \text{(download File exchange)}$$

This takes long for big systems

turns out we can find basis easily.

BC's set $h_1 = h_8 = 0$

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

first & last row of Identity

$$\underline{\underline{I}} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

Remaining unknowns $h_2 \ldots h_7$

$\Rightarrow$ basis is the 2^{nd} to 7^{th} column of $\underline{\underline{I}}$

$$\underline{\underline{N}} = \begin{bmatrix} 0 & 0 & 0 & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ n_1 & n_2 & n_3 & \cdots & n_7 \\ | & | & | & & | \end{bmatrix}$$

* $\underline{\underline{N}}\,\underline{h}$ is always zero on bnd.

$\Rightarrow$ see live script!

Heterogeneous BC

$$\boxed{\underline{\underline{B}}\,\underline{h} = g} \qquad g = \begin{bmatrix} h_D \\ h_T \end{bmatrix}$$

$\underline{\underline{B}}$ is same as before
just specifies location of BC

because $\underline{\underline{B}}\,\underline{h} = g$ is linear we decompose

$$\underline{h} = \underline{h}_0 + \underline{h}_p$$

homogeneous: $\underline{\underline{B}}\,\underline{h}_0 = \underline{0}$

heterogeneous: $\underline{\underline{B}}\,\underline{h}_p = g$

$$\underline{\underline{B}}\,\underbrace{(\underline{h}_0 + \underline{h}_p)}_{\underline{h}} = g$$

Note: $\underline{h}$ is unique

but split $h = \underline{h}_0 + \underline{h}_p$ is not unique but there is obvious simplest choice

Two questions:

1) how do we find $\underline{h}_p$?

2) Given $\underline{h}_p$ how do we find the associated $\underline{h}_0$?

Start with 2: Suppose we know $\underline{h}_p$

$$\underline{\underline{L}}(\underline{h}_0 + \underline{h}_p) = \underline{f}_s$$

$$\underline{\underline{L}}\,\underline{h}_0 = \underline{f}_s - \underbrace{\underline{\underline{L}}\,\underline{h}_p}_{f_D} = \underline{f}_s + \underline{f}_D \qquad \underline{f}_D = -\underline{\underline{L}}\,\underline{h}_p$$

$$\underline{\underline{L}}\,h_0 = \underline{f}_s + \underline{f}_D = \underline{f}$$

$\Rightarrow$ reduced system $\qquad \underline{\underline{L}}_r\, h_{0,r} = f_r$

$$\underline{h}_0 = \underline{N}\, h_{0,r}$$

$$f_r = \underline{\underline{N}}\,\underline{f} = \underline{\underline{N}}(\underline{f}_s + \underline{f}_D)$$

But how do we find $\underline{h}_p$?

Note that h_p does not need to satisfy

$$\underline{\underline{L}}\, \underline{h}_p \neq \underline{f}_s$$

It just needs to satisfy $\underline{\underline{B}}\, h_p = \underline{g}$

Simple solution: $h_p = \begin{bmatrix} h_D \\ \vdots \\ h_T \end{bmatrix}$

This works for our case

but a more general way to find $\underline{h}_p$ is to solve another reduced system

$$\underline{h}_{pr} = \underline{\underline{B}}\, \underline{h}_p \quad \longrightarrow \quad \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$$

$$\Rightarrow \quad \underbrace{\underline{\underline{B}}\,\underline{\underline{B}}^T}_{Ne \cdot Ne} \underline{h}_{pr} = \underline{g}$$

Summary of BC implementation:

1) Find $\underline{h}_p$

$$\underline{\underline{B}}\,\underline{\underline{B}}^T \underline{h}_{pr} = g \quad \Rightarrow \quad \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$$

2) Find associated hom. solution

$$\underline{\underline{L}}_r h_{or} = f_r$$

$$\underline{h}_o = \underline{\underline{N}}\, h_{or}$$

3) Full solution $\quad \underline{h} = \underline{h}_o + \underline{h}_p$