Lecture 7: Effective Conductivity of Layered Media

Logistics: HW2 due Thursday P1: 7/12

P2: 3/12

P3: 0/12

Note: Problem 3 is updated

Last time: • Dirichlet BC & Constraints

head/unknown is prescribed on bnd

• PDE: $\boxed{\begin{array}{l} \underline{\underline{L}}\,\underline{h} = \underline{f}_s \\ \underline{\underline{B}}\,\underline{h} = \underline{g} \end{array}}$ $\underline{\underline{L}}$ system matrix

BC: $\underline{\underline{B}}$ constraint matrix

$\Rightarrow$ eliminate unknowns

• solve a reduced system: $\boxed{\underline{\underline{L}}_r \underline{h}_r = \underline{f}_{s,r}}$

$\Rightarrow \underline{\underline{L}}_r = \underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}, \quad \underline{h}_r = \underline{\underline{N}}^T \underline{h} \quad \underline{f}_{sr} = \underline{\underline{N}}^T \underline{f}_s$

$\underline{\underline{N}}$ is orthonormal basis for null space of $\underline{\underline{B}}$

• Solution strategy: $\underline{h} = \underline{h}_o + \underline{h}_p$ Problem 2

1) $\boxed{\underline{\underline{B}}\,\underline{\underline{B}}^T} \underline{h}_{pr} = \underline{g} \rightarrow \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$

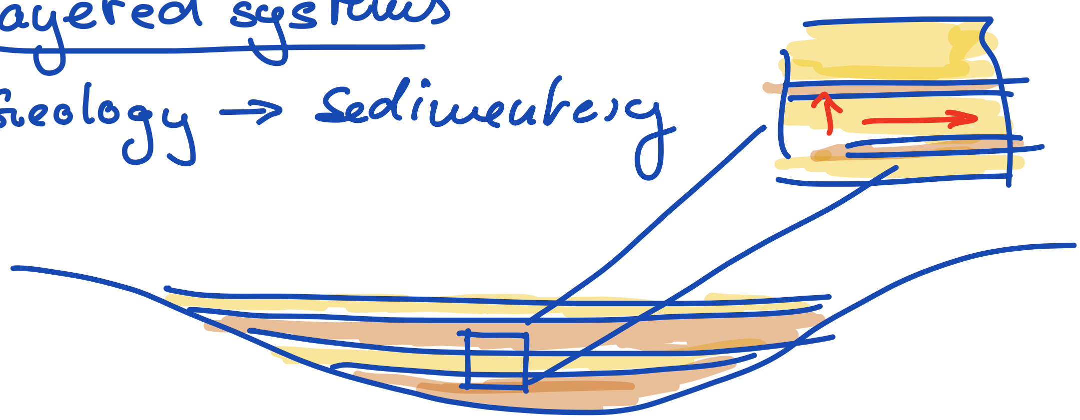
2) $\underline{\underline{L}}\,\underline{h}_o = \underline{f}_s - \underline{\underline{L}}\,\underline{h}_p = \underline{f} \Rightarrow \underline{\underline{L}}_r \underline{h}_{or} = \underline{f}_r$

3) $\underline{h} = \underline{h}_o + \underline{h}_p$ $\underline{h}_o = \underline{\underline{N}}\,\underline{h}_{or}$

Today: Layered media

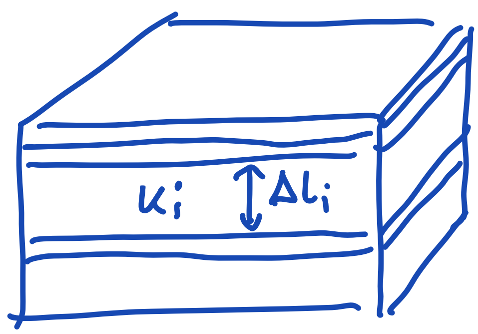
Layered systems

Geology → Sedimentary



Geologic media are layered on all scales
The smaller scales cannot be resolved in numerical model!

Layer cake:



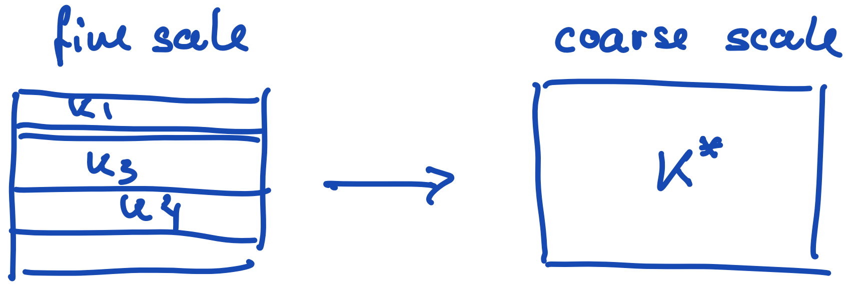
Stack of N layers with thickness ΔL_i and conductivity K_i

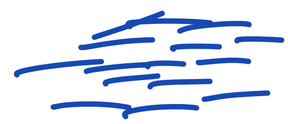
$$\sum_{i=1}^{N} \Delta L_i = \Delta L$$

Can look at two limiting cases:

1) Flow along layers (parallel)
2) Flow across layers (perpendicular)

To understand the effect of layering on flow we try to find effitive K^* that represents the entire stack (upsaling)





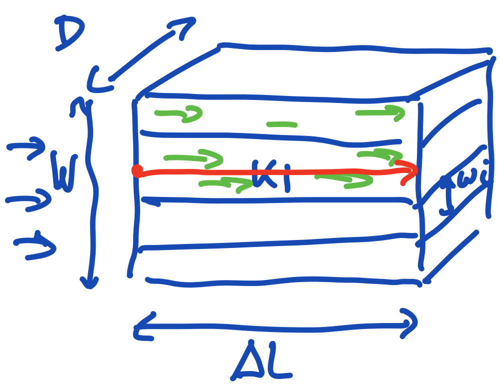
$\Rightarrow$ K^* will depend on flow direction

Fine scale : $K = K(\underline{x})$ heterogeneous (changes with location)

Coarse scale: K^* anisotropic (changes with dir.)

($\Rightarrow$ K is a tensor)

1) <u>Flow along Layers</u>



Top & bottom are impermeable

Apply Δh across sample

$\Delta h = h_R - h_L$

$\Rightarrow$ flow is 1D along each layer

$\Rightarrow$ consider each layer seperately

each layer has same $\frac{\Delta h}{\Delta L}$

Darcy's law in ith layer: $Q_i = - D \underbrace{w_i}_{A_i} K_i \frac{\Delta h}{\Delta L}$

Darcy's law for whole stack: $Q = - \underbrace{D W}_{A} K_{\parallel}^{*} \frac{\Delta h}{\Delta L}$ unknown

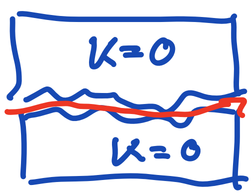
$\underbrace{Q = \sum_{i=1}^{N} Q_i}_{\text{mass cons.}}$ $\qquad W = \sum_{i=1}^{N} w_i$

$$Q = \sum_{i=1}^{N} Q_i = \sum_{i=1}^{N} - D w_i K_i \frac{\Delta h}{\Delta L} = - D \frac{\Delta h}{\Delta L} \sum_{i=1}^{N} w_i K_i$$

$$\cancel{-D} W K_{\parallel}^{*} \cancel{\frac{\Delta h}{\Delta L}} = \cancel{-D} \cancel{\frac{\Delta h}{\Delta L}} \sum_{i=1}^{N} w_i K_i$$

$$\Rightarrow \quad \boxed{K_{\parallel}^{*} = \sum_{i=1}^{N} \frac{w_i}{W} K_i}$$ weighted arithmetic average.

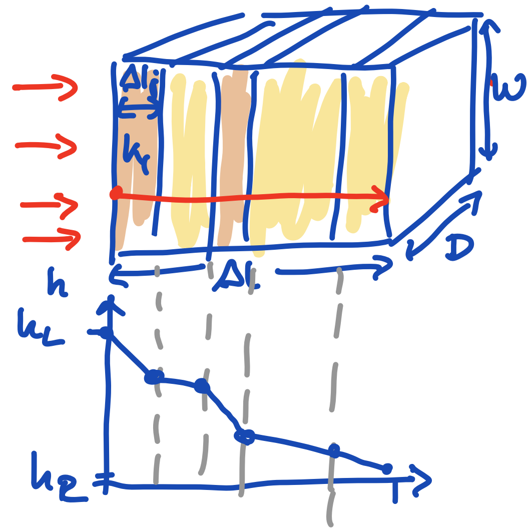
$\Rightarrow$ high K layers dominate



fracture

most of rock is impermeable but due to single high-K fracture $K_{\parallel}^{*}$ is high!

2) Flow across layers



$\Delta h = h_R - h_L$

$A = WD$ is same for each layer $\Rightarrow q$

Darcy i-th layer: $q_i = -K_i \frac{\Delta h_i}{\Delta l_i}$

$q_i = q$ same

$q = -K_i \frac{\Delta h_i}{\Delta l_i}$

head is piece wise linear

head drop $\frac{\Delta h_i}{\Delta l_i}$ is large in low-k layers

and small in high-k layers

$q = -K_1 \frac{\Delta h_1}{\Delta l_1} = -K_2 \frac{\Delta h_2}{\Delta l_2}$ $\qquad \frac{\Delta h_1}{\Delta l_1} = \frac{K_2}{K_1} \frac{\Delta h_2}{\Delta l_2}$

fix

Darcy for whole stack: $q = -K_\perp^* \frac{\Delta h}{\Delta L}$

unknown

$\Delta h = \sum_{i=1}^{N} \Delta h_i \qquad \Delta L = \sum_{i=1}^{N} \Delta l_i$

$\Delta h = \sum_{i=1}^{N} -\frac{q\,\Delta l_i}{K_i}$

$$K_\perp^* = -\frac{q\,\Delta L}{\Delta h} = \frac{\cancel{q}\,\Delta L}{\sum_{i=1}^{N} \cancel{q} \frac{\Delta l_i}{\Delta k_i}}$$

$$\boxed{K_\perp^* = \frac{1}{\sum_{i=1}^{N} \frac{\Delta L_i/\Delta L}{K_i}}}$$

Effective conductivity for flow across layers

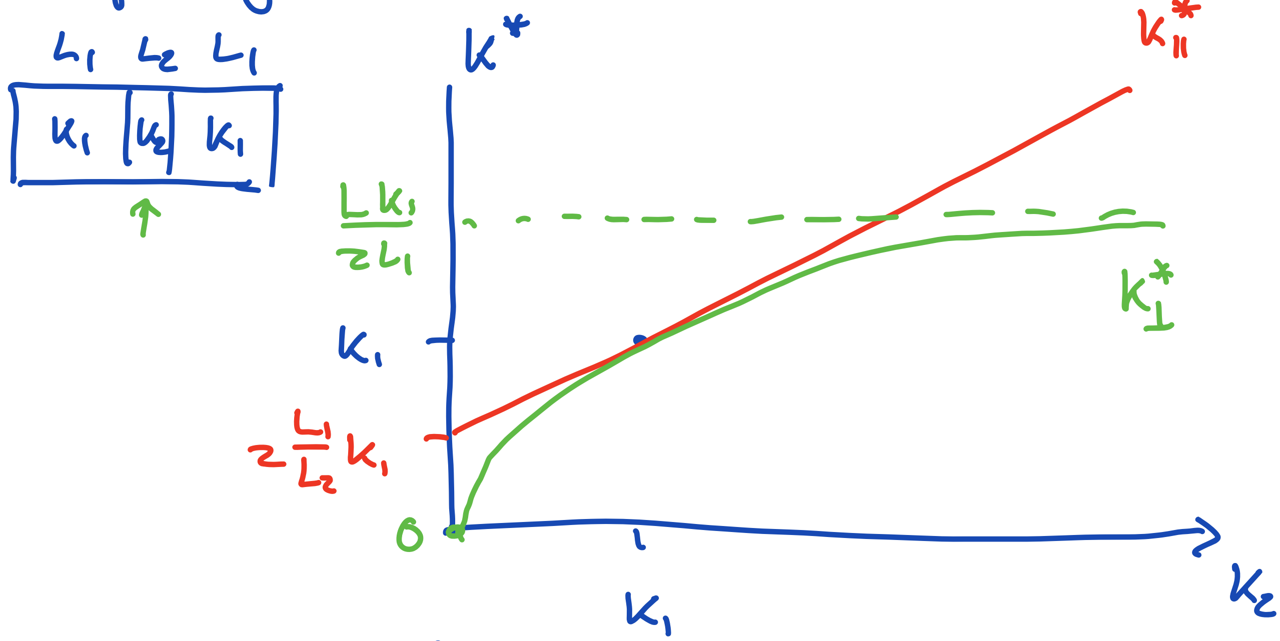
⇒ harmonic average
weighted

⇒ lowest K layer will dominate

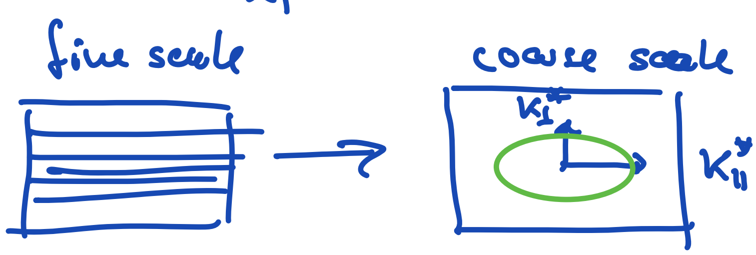
$K_\perp^* = 0$ if any $k_i = 0$

Compare $K_\parallel^*$ and $K_\perp^*$

3 layer system



Anisotropy:



⇒ $K = \underline{\underline{K}}$ tensor