Lecture 8: Heterogeneous Coefficients

Logistics: - HW2 due (bad weather extensions)

- HW3 is posted

Last time: - Layered media

- Upscaling 

$\Rightarrow$ effective properties

- Flow along & across layers

$$K^*_{\parallel} = \sum_{i=1}^{N} \frac{w_i}{W} K_i$$ arithmetic average

$\rightarrow$ high K

$$K^*_{\perp} = \frac{1}{\sum_{i=1}^{N} \frac{\Delta L_i / \Delta L}{K_i}}$$ harmonic average

$\rightarrow$ low K

Today: - Heterogeneous coefficients

- Radial & Spherical coordinates

Variable coefficients

Heterogeneity is key element

Continuous: $-\nabla \cdot \underline{q} = f_s$

$\underline{q} = -K(\underline{x})\nabla h$

$\left.\begin{matrix} -\nabla \cdot \underline{q} = f_s \\ \underline{q} = -K(\underline{x})\nabla h \end{matrix}\right\} -\nabla\cdot[\underline{\underline{K(x)}}\nabla h] = f_s$

$\cancel{-K(x)\nabla^2 h = f_s}$

$$-\nabla \cdot [\, K(\underline{x})\, \nabla h \,] = f_s$$

Discrete: $-\underline{\underline{D}} * [\, \underline{\underline{Kd}} * \underline{\underline{G}}\ \underline{h} \,] = \underline{f_s}$

What is size of $\underline{\underline{Kd}}$? D: $Nx \cdot (Nx+1)$; Kd: $(Nx+1)\cdot(Nx+1)$; G: $(Nx+1)\cdot Nx$

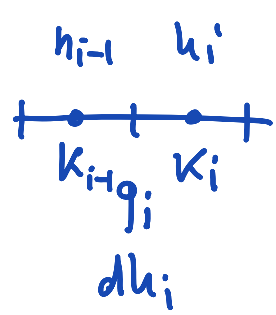
$\Rightarrow$ $\underline{\underline{Kd}}$ is $(Nx+1)$ by $Nx+1$ is associated with faces

Entries into Kd?

Darcy flux: $\underline{q} = -K \nabla h$

discrete $\underline{q} = -\underline{\underline{Kd}} * \underbrace{\underline{\underline{G}}\ \underline{\underline{h}}}_{\underline{dh}}$

$$q_i = -K_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}$$



$K_{i-\frac{1}{2}}$ is mean of k_i and k_{i-1}

$\Rightarrow$ $\underline{\underline{Kd}}$ must multiply every entry of $\underline{dh} = \underline{\underline{G}}\,\underline{h}$ with the mean of k on the interface

Kmean is vector of $(Nx+1)$ by 1 means

$$\underline{q} = -\underbrace{\underline{Kmean} \;.*\; \underline{dh}}_{\text{element wise multiplication}} \qquad q = -K \cancel{\nabla h} \frac{dh}{dx}$$

$$\underline{dh} = \underline{\underline{G}}\,\underline{h}$$

But to solve linear system for a unknown $\underline{h}$

$$\underline{q} = -\,\underline{kmean} \;.*\; \underline{\underline{G}}\,\underline{h}$$

$$\underline{q} = -\,\underbrace{\underline{\underline{Kd}} \times \underline{\underline{G}}} \times \underline{h}$$

simply place kmean on the diagonal of $\underline{\underline{Kd}}$!

$$\cancel{\frac{\partial \theta}{\partial t}}^{\,0} + \nabla \cdot q = f_s \qquad \underline{\underline{Kd}} = \begin{bmatrix} \ddots & & \\ & kmean & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \underline{dh} \end{bmatrix}$$

This allows us to form our heterogeneous operator: $\underline{\underline{L}} = - \underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}$

mass balance

$\nabla \cdot q = f_s$

$q = -K(\underline{x}) \nabla h$

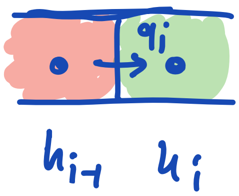
$\Rightarrow \quad \underline{\underline{L}}(h) = \underline{f_s}$

$\left[-\nabla \cdot [K(x) \nabla h] - f_s \right]$

$\underbrace{-\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}\, \underline{h}}_{\underline{\underline{L}}}$

The appropriate average depends on problem:

1) Hydraulic conductivity



flow across layer $\Big\} \Rightarrow$ harmonic average

$$\boxed{K_{i-\frac{1}{2}} = \frac{2}{\frac{1}{K_{i-1}} + \frac{1}{K_i}}}$$

$\Rightarrow$ gives correct solution in 1D

best approximation in 2D/3D

$\Rightarrow$ std. choice for hydraulic conductivity

2) Non-linear conductivity: $K = K(h)$

varies smoothly

Examples: - compressible flow (gas)

- unconfined flow

$\Rightarrow$ arithmetic. $\quad K_{i-\frac{1}{2}} = K\left(\frac{h_{i-1} + h_i}{2}\right)$

Both arithmetic and harmonic means are special cases of power-law average.

$$\boxed{K_p = \left(\frac{1}{2}\left(K_{i-1}^p + K_i^p\right)\right)^{\frac{1}{p}}}$$

p=1 arithmetic

p=-1 harmonic

$\Rightarrow$ use p to specify the type of average

Implementation:

From build_ops we have $\underline{\underline{M}}$ computes arithmetic average.

arithmetic (p=1) : $\underline{Kmean} = \underline{\underline{M}} * \underline{K}$

harmonic (p=-1) : Kmean = 1./($\underline{\underline{M}}$*(1./K))

or simply implement general power-law average

$$\boxed{\underline{Kmean} = (\underline{\underline{M}} * \underline{K}.\hat{\ }p).\hat{\ }(1/p)}$$

place Kmean on diagonal

$\underline{\underline{Kd}}$ = spdiags($\underline{Kmean}$, 0, Nx+1, Nx+1)

$\Rightarrow$ $\underline{\underline{Kd}}$=comp_mean($\underline{K}$, $\frac{-1}{p}$)