Lecture 9: Fluxes and Flux Boundary Conditions

Logistics: - HW3 is due 10/13 12

⇒ if you have problems see me !

- HW4 is posted

- Afzal is willing to TA !

Last time: - Variable coefficients $K = K(\underline{x})$

$-\nabla \cdot [K(\underline{x}) \nabla h] \approx -\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}$

$\underline{\underline{Kd}}$ diagonal with Kmean on faces

$\boxed{\text{Kmean} = (\underline{\underline{M}} * \underline{K}.\hat{\ }p).\hat{\ }(1/p)}$

$p = 1$ arithmetic → along layers

$p = -1$ harmonic → across layers

⇒ choose harmonic for conductivity

Today: - Flux computation $h \rightarrow q$

- Flux boundary conditions

Neumann/Flux Boundary Conditions

Dir. BC prescribe head/unknown on bnd

$\Rightarrow$ eliminate heads Dir. bnd

Neu. BC prescribe flux

$\Rightarrow$ still need to solve for head on Neu. bnd.

$\Rightarrow$ Neu. BC are <u>not</u> implemented as constraints.

<u>Sign convention</u>

In class we consider in flows to be <u>positive</u>

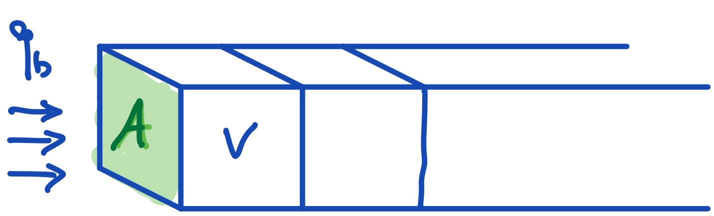
$$\boxed{q \cdot \hat{n}_i = q_b}$$

q_B = bnd flux

$\hat{n}_i$ = <u>inward</u> unit normal

$\Rightarrow$ $q_B > 0$ is an inflow

<u>Implementation of Neu. BC</u>



$\text{flux} = \frac{\#}{L^2 T}$

A = area of bnd face

V = cell volume

Implement flux BC as an equivalent source/sink term in bnd cell

Flow rate across bnd face: $Q_b = A\, q_b$

Equivalent source term: $Q_b = V f_n$

f_n = Neumann source term $\frac{L^3}{L^3 T} = \frac{1}{T}$

$$\Rightarrow \boxed{f_n = q_b \frac{A}{V}} \quad \text{for a single cell}$$

Note: sign of f_n is automatically correct because $q_b > 0$ is an flow

$\Rightarrow$ $f_n > 0$ is source

In general $\underline{f_n}$ is N_x by 1 rhs vector with N_n non-zero entries, one entry for each cell with a Neu. BC.

For a problem with Neu. BC the linear system is:

$$\underline{\underline{L}}\, \underline{h} = \underline{f}_s + \underline{f}_n$$

To build f_N we define: Nu = # of Neu BC

BC.dof_neu = Nu by 1 vector of cells with Neu BC

BC.dof_f_neu = Nu by 1 vector of faces with Neu BC

BC.qb = Nu by 1 vector of prescribed fluxes

need to add cell volumes & face ~~area~~ areas to Grid:

Grid.A = Nfx by 1

Grid.V = Nx by 1

Compute & place the Nn entries of fn:

fn = spalloc(Grid.Nx, 1, Nn)

$$\boxed{\texttt{fn(BC.dof\_neu) = qb.*Grid.A(Grid.dof\_f\_neu)./Grid.V(Grid.dof\_neu)}}$$

↑ Nu·1

⇒ ~~one~~ two line addition to build_bnd

No Nn = number of cells with Neu. BC.

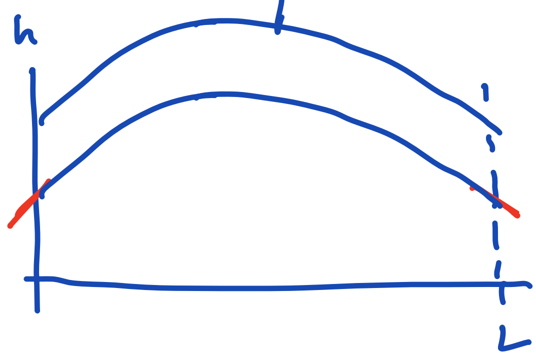
⇒ see Live Script.

Note: Cannot set pure Neumann BC.

1) Compatability

Sum of bnd fluxes must be equal to sum of source/sinks in domain otherwise there is no solution.

2) If compatability is satisfied the solution to pure Neumann problem is not unique.



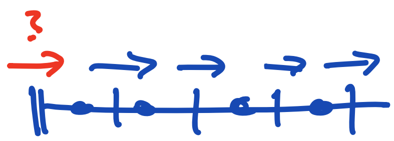
undetermined constant in head

Compute Fluxes of Gradient Fields

$q = -K \nabla h$ h = scalar potential

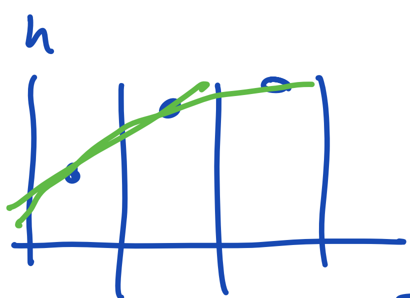
$\boxed{\underline{q} \approx -\underline{\underline{K_d}} * \underline{\underline{G}}\,\underline{h}}$ discrete approx.

This works on the interior of the domain but not on boundaries, because $\underline{\underline{G}}\,\underline{h}$ is zero by construction.



$\Rightarrow$ reconstruct flux on bndface

Option 1



use one-sided derivative to approximate $\frac{dh}{dx}$ on bnd.

Problem: loose discrete conservation

Option 2: Reconstruct bnd flux from discrete mass balance in bnd cell

Discrete linear system: $\boxed{\underline{\underline{L}}\,\underline{h} = \underline{f}_s}$

Discrete residual: $\underline{r}(\underline{h}) = \underline{\underline{L}}\,\underline{h} - \underline{f}_s$

If the discre eqns are satisfied $\underline{r} = \underline{0}$

⇒ in bnd cells $\underline{r} \neq 0$ because $\underline{\underline{G}}$ is zero

⇒ non-zero $\underline{r}$ in bnd cell s contain information about bnd fluxes!

Consider a problem with flux bnd

$$\underline{r}(\underline{h}) = \underline{\underline{L}}*\underline{h} - \underline{f}_s \quad \text{(discrete PDE)}$$

but linear system we are solving

$$\underline{\underline{L}}\,\underline{h} = \underline{f}_s + \underline{f}_n$$

$$\underline{r}(h) = \underline{f}_n$$

⇒ residuals are equal to rhs flux vector

⇒ reconstruct bnd flux from residual

Entries into $\underline{f_n}$ are : $f_n = q_b \frac{A}{V}$

Now we are given $\underline{r} = f_n \rightarrow q_b$

$$q_b = f_n \frac{V}{A} = r \frac{V}{A} \quad \text{(singel cell)}$$