Incompressible Flow

- For the pressure variations encountered during groundwater flow the density of water is nearly constant. $\Rightarrow$ $\rho(p) \approx \rho_0 = \text{const.}$
- The porosity is highly variable in space but constant in time, in absence of reactions & compaction

$$\Rightarrow \quad \phi = \phi(\underline{x})$$

$\Rightarrow$ simplification of fluid mass balance

$$\frac{\partial}{\partial t}\left(\cancel{\phi \rho_0}^{\,0}\right) + \nabla\cdot(\rho_0 \vec{q}) = \rho_0 f \qquad \Rightarrow \qquad \boxed{\nabla\cdot\vec{q} = f}$$

- Darcy's law for constant density:

$$h = z + \frac{p-p_0}{\rho_0 g} \qquad \text{"hydraulic head"} \quad [L]$$

$$K = \rho_0 g \frac{k}{\mu} \qquad \text{"hydraulic conductivity"} \quad \left[\frac{L}{T}\right] \qquad \boxed{\bar{q} = -K\nabla h}$$

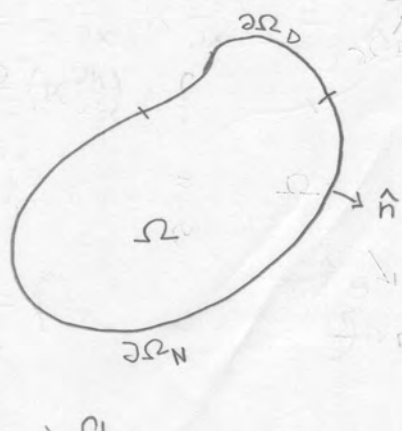
Equation for incompressible flow $\quad \boxed{-\nabla\cdot(K\nabla h) = f}$ Poisson Eqn.

Boundary Value Problem (BVP)

A well posed problem requires boundary conditions

PDE $\quad \nabla\cdot\bar{q} = f$

$\qquad\quad \bar{q} = -K\nabla h$



BC: a) Dirichlet (prescribed head)

$\qquad h = h_B(\underline{x}) \quad \underline{x} \in \partial\Omega_D$

b) Neuman (prescribed flux)

$\qquad \bar{q}\cdot\hat{n} = -q_B \quad \underline{x} \in \partial\Omega_D$

Note: $q_B > 0$ corresponds to an inflow

Basic properties of incompressible flow

1) Linearity: The Laplacian ∇^2 is a linear operator

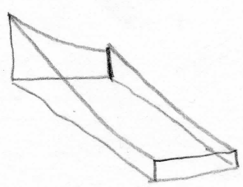
$$\nabla^2(c_1 h_1 + c_2 h_2) = c_1 \nabla^2 h_1 + c_2 \nabla^2 h_2$$

$\Rightarrow$ Principle of superposition

If h_1 and h_2 are solutions than arbitrary combinations are also solutions.

Note: source terms and BC's add up.

2) Maximum principle:



If $r=0$ the solution cannot attain its maximum in the interior of the domain (unless it is a constant).

$\Rightarrow$ maximum is on boundary

3) Solution is unique & problem is well posed

- If a solution exists it is unique
- Solution changes continuously with the data

  i.e.) small change in BC's $\Rightarrow$ small change in soln.

4) Solvability condition (Compatability cond.)

For problems with pure Neuman BC's solutions only exist if

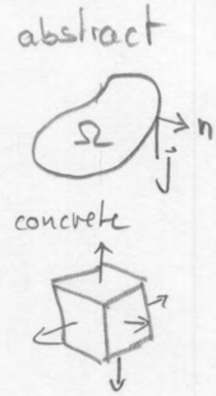
$$\boxed{\oint_{\partial\Omega} q \cdot \hat{n}\, ds = \int_{\Omega} r\, dV}$$

Physical Interpretation:

At steady state all fluid produced in Ω by r must exit Ω at $\partial\Omega$!

$\Rightarrow$ common cause of numerical problems.

Note on curvilinear coordinates



The derivation of balance laws in terms of div & grad is independent of coordinate system used? Benefit of abstract derivation over a derivation considering a concrete domain such as a box?

⇒ Equations written in div & grad are independent of the coordinate system

General $\quad -\nabla \cdot K \nabla h = f$

Cartesian $\quad -\frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z}\right) = f$

Cylindrical. $\quad -\frac{1}{r}\frac{\partial}{\partial r}\left(r K \frac{\partial h}{\partial r}\right) - \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(K \frac{\partial h}{\partial \theta}\right) - \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z}\right) = f$

Spherical : $\quad -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K \frac{\partial h}{\partial r}\right) - \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \, K \frac{\partial h}{\partial \theta}\right) - \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(K \frac{\partial h}{\partial \phi}\right) = f$

⇒ div-grad notation is compact and hides the complexities of the non-cartesian coordinate systems

⇒ we want the same for numerical implementation