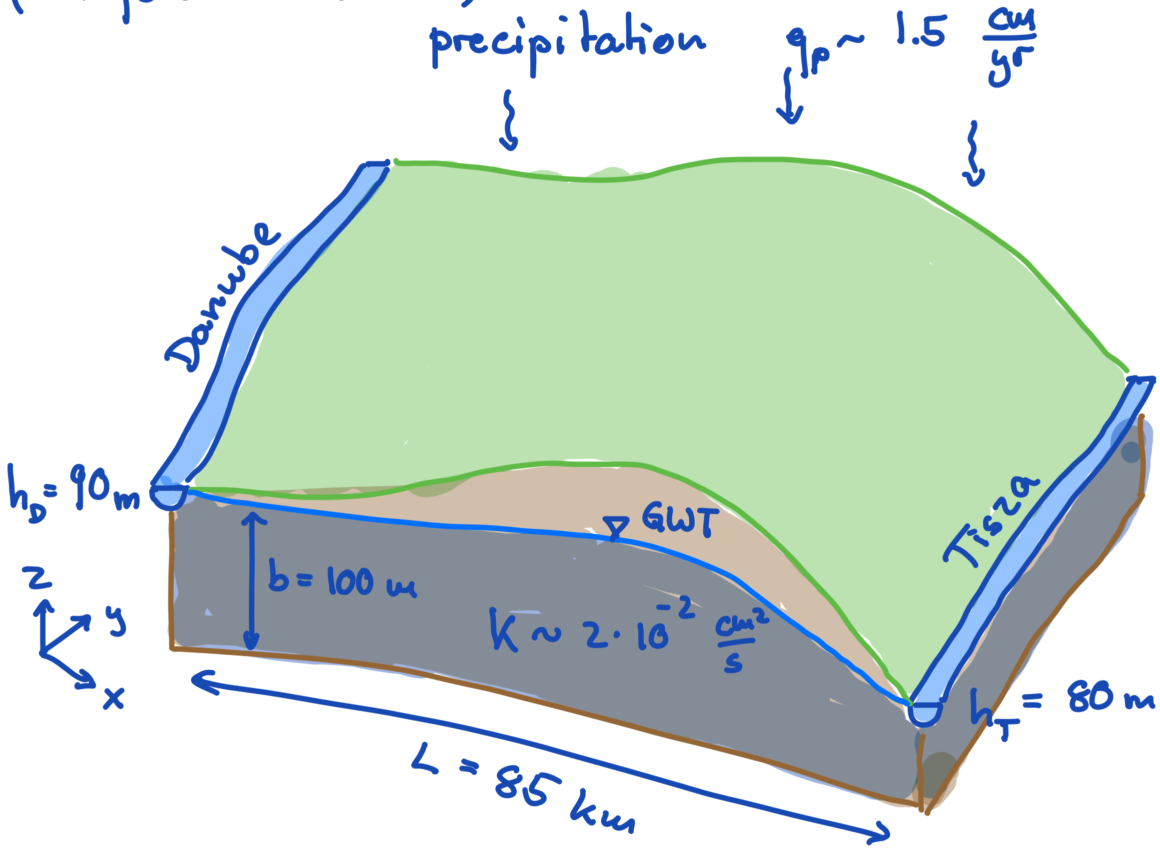
Groundwater recharge between two rivers

(Sanford et al. 2001)



Aquifer aspect ratio: $b/L = \frac{100}{85\,000} = \frac{1}{850} \sim 0.001$

$\Rightarrow$ flow is practically 1D in horizontal direction

This can be seen from a scaling analysis of the continuity equation.

Introduce characteristic scales:

$$x_D = \frac{x}{L} \qquad z_D = \frac{z}{b} \qquad q_{x,D} = \frac{q_x}{q_{x,c}} \qquad q_{z,D} = \frac{q_z}{q_{z,c}}$$

substitute into continuity, eg.. $q_x = q_{x,c} \; q_{x,D}$

$$\nabla \cdot q = \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = \frac{q_{x,c}}{L}\,\frac{\partial q_{x,D}}{\partial x_D} + \frac{q_{z,c}}{b}\,\frac{\partial q_{z,D}}{\partial z_D} = 0$$

collect terms

$$\frac{\partial q_{x,D}}{\partial x_D} + \underbrace{\frac{q_{z,c}\,L}{q_{x,c}\,b}}_{\Pi}\,\frac{\partial q_{z,D}}{\partial z_{,D}} = 0$$

Π dimensionless parameter

Set $\Pi = 1$ to get relation between fluxes

$$\boxed{q_{z,c} = \frac{b}{L}\, q_{x,c} \ll q_{x,c}}$$

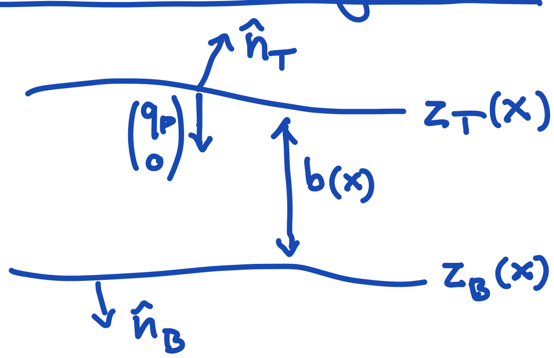
$\Rightarrow$ vertical flux is negligible

Assume $q_z = 0 \Rightarrow \frac{\partial h}{\partial z} = 0 \Rightarrow h(x)$

Darcy: $q_h = \begin{pmatrix} q_x \\ q_y \end{pmatrix} = -K\,\nabla_h h \qquad \nabla_h h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix}$

$\uparrow$ horizontal gradient

Vertical integration



$b(x) = z_T(x) - z_B(x)$

$$\int_{z_B(x)}^{z_T(x)} \nabla \cdot q \, dz = ?$$

Need to exchange integral and derivative
but z_T & z_B depend on x !

Standard Leibnitz Integral Rule

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,z)\,dz = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\,dz + f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx}$$

Here we need slightly different form: (Pinder & Gray)

$$\int_{z_B}^{z_T} \nabla\cdot q\,dz = \nabla_h\cdot\int_{z_B}^{z_T} q_h\,dz + (\hat{n}\cdot q|_{z_T} - (\hat{n}\cdot q|_{z_B}$$

note: $\nabla\cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ $\qquad \nabla_h\cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$$q = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} \qquad q_h = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

since $\quad q_x \neq q_x(z) \qquad q_y \neq q_y(z)$

$$\Rightarrow \nabla_h\cdot\int_{z_B}^{z_T} q_h\,dz = \nabla_h\cdot\left(q_h\int_{z_B}^{z_T} dz\right) = \nabla_h\cdot(b\,q_h)$$

Boundary terms:

bottom: assume an impermeable base

$$\Rightarrow (\hat{n}\cdot q|_{z_B} = 0$$

top: assume $q|_{z_T} = \begin{pmatrix} 0 \\ 0 \\ -q_p \end{pmatrix}$ $\qquad \hat{n} \approx \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad (\hat{n} \cdot q|_{z_T} = -q_p$$

Substitute: $\nabla_h \cdot (b\, q_h) - q_p = 0$

$$\Rightarrow \quad \boxed{-\nabla_h \cdot (b K \nabla_h h) = q_p} \qquad \text{2D Shallow Aquifer}$$

Note: In unconfined aquifer $b = h$ $\Rightarrow$ non-linear

here we assume a confined aquifer

In 1D we have: $\boxed{-\frac{\partial}{\partial x}\left[b K \frac{\partial h}{\partial x}\right] = q_p}$

Simplified example problem:

PDE: $-\frac{d}{dx}\left[b K \frac{dh}{dx}\right] = q_p \qquad x \in [0, L]$

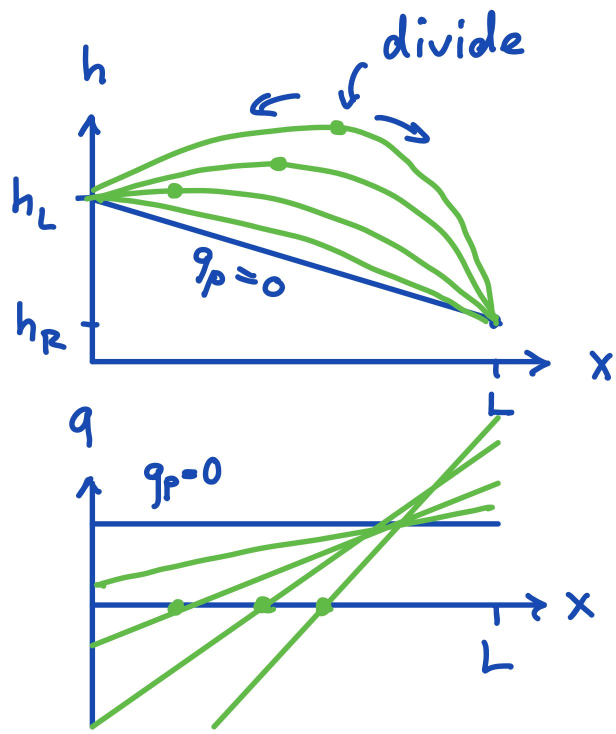
BC: $h(0) = h_D \qquad h(L) = h_T$

Integrate twice to obtain analytic solution

$$h = h_D + \left(\frac{h_T - h_D}{L} + \frac{q_p L}{2bK} \right) x - \frac{q_p}{2bK} x^2$$

$$q = \frac{q_p}{b} \left(x - \frac{L}{2} \right) - \frac{K}{L} \left(h_T - h_D \right)$$

Sketch of solution:



As recharge increases a "groundwater divide" forms that separates water flowing to the Danube and the Tisza rivers.

$\Rightarrow$ solve numerically!