Heat conduction

General energy conservation:

$$\overline{\rho c_p} \frac{\partial T}{\partial t} + \nabla \cdot [q \rho_f c_{p,f} T - \bar{\kappa} \nabla T] = \hat{f}_s$$

$$\overline{\rho c_p} = \phi \rho_f c_{p,f} + (1-\phi) \rho_s c_{p,s}$$

$$\bar{\kappa} = \phi \kappa_f + (1-\phi) \kappa_s$$

Consider the limit of pure rock

$\phi = 0$, $q = 0$ and $\rho_s \equiv \rho$ $\kappa_s \equiv \kappa$ $c_{p,s} \equiv c_p$

radiogenic source term:

$\hat{f}_s = \rho H_0$ H = heat production $\left[\frac{W}{kg}\right]$

$\Rightarrow$ Heat equation

$$\boxed{\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot [\kappa \nabla T] = \rho H}$$

- Transient $\rightarrow$ changes with time
- radiogenic source term

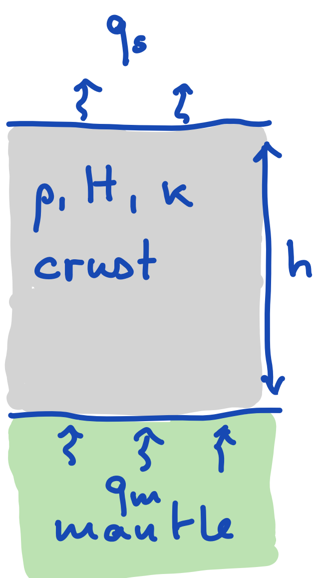
Steady heat conduction - Crustal Geotherm

Stead → no change in time

$$\Rightarrow \quad \boxed{-\nabla\cdot[k\nabla T] = \rho H}$$

Same equation we have solved for ground-water flow, except for source term.

⇒ know how to solve !



q_s = surface heat flow

q_m = mantle heat flow

h = ave. thickness of cont. crust

We know appoximately:

$$q_s = 65\cdot 10^{-3}\ \frac{W}{m^2}$$

$$\rho = 2700\ \frac{kg}{m^3}$$

$$h = 35\ km$$

$$k = 3.35\ \frac{W}{m\,K}$$

$$H = 9.6\cdot 10^{-10}\ \frac{W}{kg} \quad \text{(at surface)}$$

We don't know q_m.

Assuming $q_m = 0$

$$q_s = \int_0^h \rho H \, dz = \rho H h = 90.72 \cdot 10^{-3} \ \frac{W}{m^2}$$

bigger than measured value

Because other quantities are reasonably well known

$\Rightarrow$ H at surface is not representative

$$\boxed{H(z) = \rho H_0 \, e^{-z/h_r}}$$

H_0 = surface heat production

h_r = decay depth of radiogenic heating

z = depth below surface

Following 1D heat conduction problem:

$$\boxed{\begin{array}{ll} \text{PDE}: -\kappa \dfrac{d^2T}{dz^2} = \rho H_0 \, e^{-z/h_r} & z \in [0, h] \\ \text{BC}: \quad T(z=0) = T_s \ , & \kappa \dfrac{dT}{dz} = q_m \end{array}}$$

Here q_m and h_r are unknown

Note: Because z points down upward heat flows are negative!

Relation between q_s and q_m

$$\frac{dq}{dz} = \rho H_0 e^{-z/h_r}$$

integrate

$$q_m - q_s = \rho H_0 \int_0^h e^{-z/h_r} dz$$

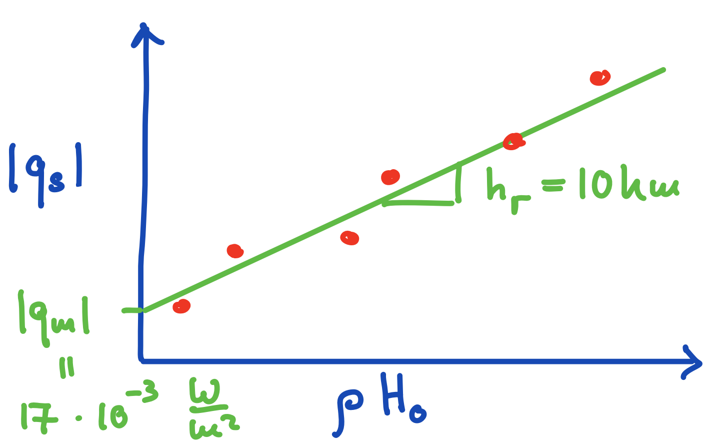
$$= -\rho h_r H_0 \left(\underbrace{e^{-h/h_r}}_{\approx 0} - \underbrace{e^{0}}_{1} \right)$$

$$q_m - q_s \approx \rho h_r H_0$$

$$q_m = -|q_m| \qquad q_s = -|q_s|$$

$$-|q_m| + |q_s| \approx \rho h_r H_0$$

$$\Rightarrow \boxed{|q_s| \approx |q_m| + h_r \rho H_0}$$

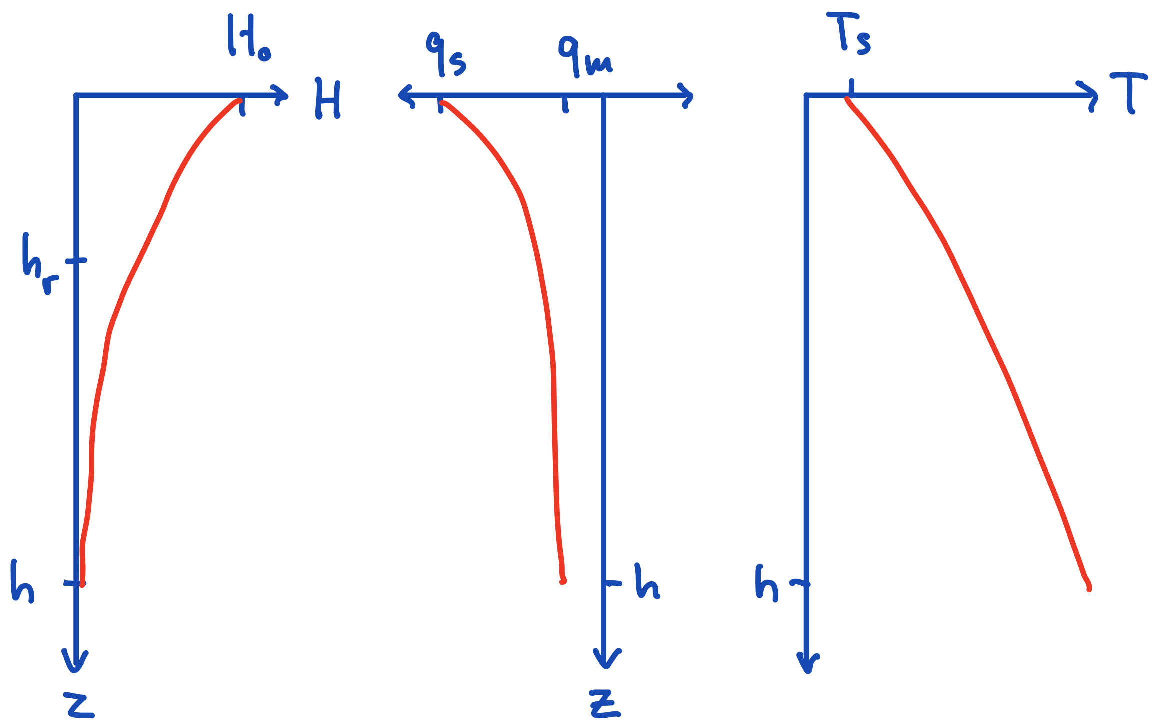


Provides a way to estimate unknown q_m and h_r from surface data!

Analytic solution for geotherm

Integrating twice

$$\boxed{\begin{aligned} q(z) &= -|q_m| - \rho H_0 h_r \left(e^{-z/h_r} - e^{-h/h_r}\right) \\ T(z) &= T_0 + \left(\frac{|q_m|}{k} - \frac{\rho H_0 h_r}{k} e^{-h/h_r}\right) + \frac{\rho H_0 h_r^2}{k}\left(1 - e^{-z/h_r}\right) \end{aligned}}$$



Implementation:

Continuous: $-\nabla \cdot \kappa \nabla T$

Discrete: $\underline{\underline{L}} = -\underline{\underline{D}}\,\underline{\underline{Kd}}\,\underline{\underline{G}}$

solve: $\underline{\underline{L}}\,\underline{u} = \underline{f}_s$ $\qquad \underline{u} = T(\underline{x})$

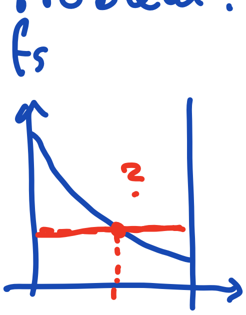
What about source term?

Continuous: $f_s = \rho H_0 e^{-x/h_r}$

Discrete $\underline{f}_s \overset{?}{=} f_s(\underline{x}_c)$

This will converge to correct answer with mesh refinement, but on coarse grid we have errors in surface heat flux.

Problem: We simply evaluate source term in cell center → may not be good representation of average heat production in the cell.



We can directly compute average over a cell

$$\langle f_{s,i} \rangle = \frac{1}{\Delta x} \int_{x_{f,i}}^{x_{f,i+1}} \rho H_0 e^{-x/h_r} dx =$$

$$\boxed{\langle f_{s,i} \rangle = \frac{\rho H_0 h_r}{\Delta x} \left[\exp\left(-\frac{x_{f,i}}{h_r}\right) - \exp\left(-\frac{x_{f,i+1}}{h_r}\right) \right]}$$

$\Rightarrow$ ensure that the correct amount of heat is added even on course grid!