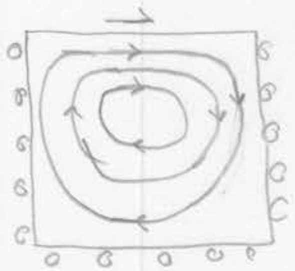
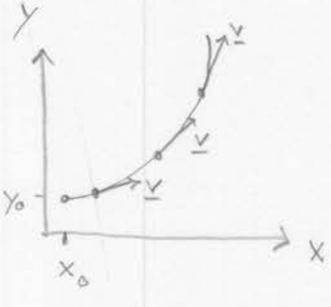
Streamlines & Stream function



Streamlines provide one of the best ways to illustrate flow fields, if applicable.

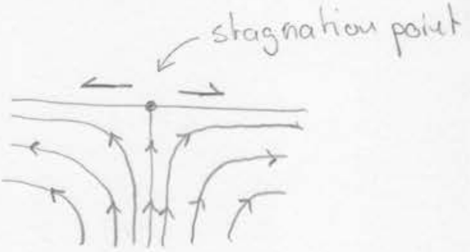
Definition:
Streamlines are the family of curves that are instantaneously tangent to the velocity field.



The definition of velocity provides a system of ODE's to compute streamlines:

$$\left.\begin{aligned} 1)\ \frac{dx}{dt} &= v_x(\underline{x}) \\ 2)\ \frac{dy}{dt} &= v_y(y) \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{V_y}{V_x} \qquad \text{where} \quad \underline{v}(\underline{x}) = \begin{pmatrix} v_x(\underline{x}) \\ v_y(\underline{x}) \end{pmatrix}$$

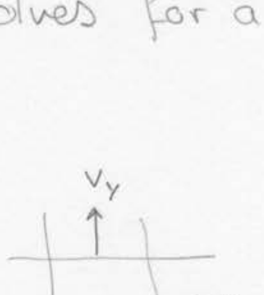
Notes:
- Safer to solve the system of ODE's because $\frac{dy}{dx}$ may not be bounded and $y(x)$ may be multivalued, as in our example.
- But the ODE's system has problems with stagnation points because $|\underline{v}| \to 0$ and $t \to \infty$



- Can only determine stagnation points by trial & error.

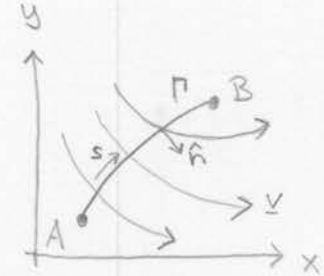
In Matlab the function streamline.m solves for a streamline given an initial point.

Note: streamline.m needs all velocities at the same location, so you have to interpolate $\underline{v}$ to the cell centers.



$\Rightarrow$ introduce errors: streamlines hitting no flow bnd's

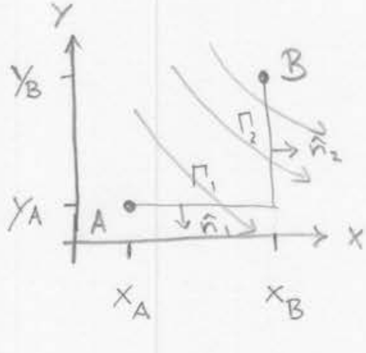
Different way of thinking about streamlines.



Compute the cummulative flux between A & B.

$$\boxed{\psi = \int_{\Gamma} \underline{v} \cdot \hat{n}\, ds}$$

Γ is the path
s is an arclength variable along Γ
$\hat{n}$ is the right hand normal



In the absence of fluid sources ψ should not depend on path. This will be confirmed below.

$\Rightarrow$ Choose a path that simplifies integration.
We break Γ into two parts along coord. directions.

along Γ_1: $\underline{v} \cdot \hat{n}_1 = -v_y$ (because $\hat{n}$ points in $-y$ dir.)

along Γ_2: $\underline{v} \cdot \hat{n}_2 = v_x$

Hence we write integral as: $\psi = \underbrace{\int_{x_A}^{x_B} -v_y(x, y_A)\, dx}_{\Gamma_1} + \underbrace{\int_{y_A}^{y_B} v_x(x_B, y)\, dy}_{\Gamma_2}$

Suppose: $y_A = y_B$

$$\psi = \int_{x_A}^{x_B} -v_y\, dx \overset{\text{F.T.C.}}{=} \int_{x_A}^{x_B} \frac{\partial \psi}{\partial x}\, dx \Rightarrow \frac{\partial \psi}{\partial x} = -v_y$$

Suppose: $x_A = x_B$

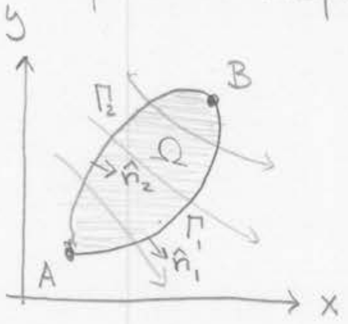
$$\psi = \int_{y_A}^{y_B} v_x\, dx \overset{\text{F.T.C}}{=} \int_{y_A}^{y_B} \frac{\partial \psi}{\partial y}\, dy \Rightarrow \frac{\partial \psi}{\partial y} = v_x$$

Therefore: $\boxed{\frac{\partial \psi}{\partial x} = -v_y \quad \frac{\partial \psi}{\partial y} = v_x}$ This is often given as the definition of the stream function

Physical Interpretation:

- Change of cummulative flux in x-dir is proportional to the negative velocity in y-dir.
- Change of cummulative flux in y-dir is proportional to the velocity in the x-dir.

The conclusions above hold if the integral defining ψ is path independent.



$$\int_{\Gamma_1} \underline{v}\cdot\hat{n}_1\, ds = \int_{\Gamma_2} \underline{v}\cdot\hat{n}_2 ds \Rightarrow \int_{\Gamma_1} \underline{v}\cdot\hat{n}_1 ds - \int_{\Gamma_2} \underline{v}\cdot\hat{n}_2 ds = 0$$

Combine paths $\Gamma = \Gamma_1 + \Gamma_2$ and define outside normal, $\hat{n}$, to the enclosed area Ω.

note that $\hat{n} = \hat{n}_1$ on Γ_1 but $\hat{n} = -\hat{n}_2$ on Γ_2

$$\int_{\Gamma_1} \underline{v}\cdot\hat{n}\, ds + \int_{\Gamma_2} \underline{v}\cdot\hat{n}\, ds = \oint_{\Gamma} \underline{v}\cdot\hat{n}\, ds = 0$$

Hence the Integral is path independent if $\oint_{\Gamma} \underline{v}\cdot\hat{n}\, ds = 0$

Using the divergence theorem: $\oint_{\Gamma} \underline{v}\cdot\hat{n}\, ds = \int_{\Omega} \nabla\cdot\underline{v}\, dV = 0$

Hence the stream function is well defined, if $\boxed{\nabla\cdot\underline{v} = 0}$

- flow is <u>in</u>compressible
- no sources/sinks of mass

} not problem for us

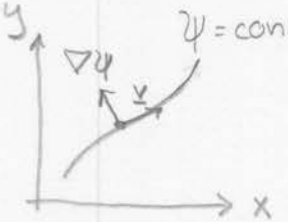
Note: In 3D there are two stream functions.

In an incompressible flow without sources the cummulative flux ψ is a unique function of $\underline{x}$ and called the stream function.

What is the relation between ψ and streamlines?

Relation between ψ and streamlines

1) The level sets of ψ are tangential to the velocity vector.

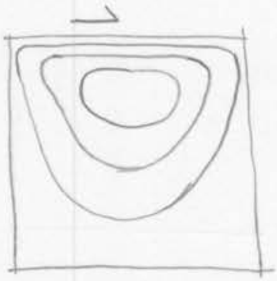


$$\nabla\psi \cdot \underline{v} = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)\cdot\begin{pmatrix}v_x\\v_y\end{pmatrix} = (-v_y \;\; v_x)\cdot\begin{pmatrix}v_x\\v_y\end{pmatrix} = -v_y v_x + v_x v_y = 0$$

$$\boxed{\nabla\psi\cdot\underline{v}=0} \quad\Rightarrow\quad \nabla\psi \perp \underline{v}$$

Level sets of ψ are the streamlines.

2)



The magnitude of the velocity is equal to the magnitude of. ∇ψ.

$$|\nabla\psi| = \sqrt{(-v_y)^2 + v_x^2} = \sqrt{v_x^2+v_y^2} = |\underline{v}|$$

If we plot equally spaced contours of ψ the spacing indicates the velocity.

This is the most useful aspect of properly plotted streamlines! Not possible without stream function.

Other applications:

We can compute a ψ for any solenoidal vector field ($\nabla\cdot\underline{a}=0$)

For example we have used it to visualize the conductive heat flux near a conductive inclusion. However, heat production by decay heating prevents computation of ψ in many cases.