Variable coefficients

Heterogeneity is a key element of porous media

Continuous eqn.: $\qquad -\nabla \cdot [K(x) \nabla h] = f_s$

Discrete eqn.: $\qquad -\underline{\underline{D}} * [\underline{\underline{Kd}} * \underline{\underline{G}}\, \underline{h}] = \underline{f_s}$

Size of $\underline{\underline{Kd}}$ matrix?

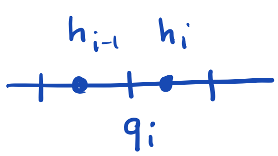
$\underline{\underline{D}}$	$\underline{\underline{Kd}}$	$\underline{\underline{G}}$
$Nx \cdot (Nx+1)$	$(Nx+1) \cdot (Nx+1)$	$(Nx+1) \cdot Nx$

$\Rightarrow$ $\underline{\underline{Kd}}$ is $(Nx+1)$ by $(Nx+1)$ matrix associated with faces

Entries into $\underline{\underline{Kd}}$ matrix?

Darcy flux: $\quad q = -K \nabla h$

$$\underline{q} = -\underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}$$



$$q_i = -K_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}$$

where $K_{i-\frac{1}{2}}$ is mean of K_{i-1} and K_i

$\Rightarrow$ $\underline{\underline{Kd}}$ must multiply every entry of $\underline{\underline{G}} * \underline{h}$ with the mean of K on interface if we had an $(Nx+1)\cdot Nx$ vector $\underline{Kmean}$

we could simply write: $q = -\underline{Kmean} .* \underbrace{\underline{\underline{G}} * h}_{\underline{dh}}$

element wise multiplication

But to form $\underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}$ we need to write

$\underline{Kmean} .* \underline{dh}$ as $\underline{\underline{Kd}} * \underline{dh}$.

$\Rightarrow$ simply place $\underline{Kmean}$ on diagonal of $\underline{\underline{Kd}}$.

The appropriate average depends on problem:

1) Hydraulic conductivity $\rightarrow$ harmonic mean

because $K(x)$ is often discontinuous (layering)

$\Rightarrow$ flow across layers (from one cell into the next)

$$\boxed{K_{i-1/2} = \frac{2}{\frac{1}{K_{i-1}} + \frac{1}{K_i}}} \qquad (\Delta l_{i-1} = \Delta l_i) \qquad \text{harmonic mean}$$

- gives correct solution in 1D

best approximation in higher dimensions

$\Rightarrow$ standard choice for hyd. conductivity.

2) Non-linear conductivity: $K = K(h)$

Examples: - compressible flow (gas)

- unconfined flow

Since h is smooth $K(h)$ is typically smooth

$\Rightarrow$ arithmetic mean is typically best

There are two options:

I) Evaluate then average

$$\boxed{K_{i-\frac{1}{2}} = \frac{K(h_{i-1}) + K(h_i)}{2}}$$

II) Average then evaluate

$$\boxed{K_{i-\frac{1}{2}} = K\left(\frac{h_{i-1}+h_i}{2}\right)}$$

Note: Arithmetic and harmonic means are special cases of a general power-law average.

$$K_p = \left(\frac{1}{2}\left(K_{i-1}^p + K_i^p\right)\right)^{\frac{1}{p}}$$

$p = 1$ is arithmetic

$p = -1$ is harmonic

$\Rightarrow$ we use p in the code to identify average!

Implementation in comp_mean.m

From build_ops we have $\underline{\underline{M}}$ matrix that "averages" from cell center to cell faces.

This is an arithmetic average!

As such we can compute averages as follows

arithmetic (p=1): $\underline{\text{Kmean}} = \underline{\underline{M}} * \underline{K}$

harmonic (p=-1): $\underline{\text{Kmean}} = 1./(\underline{\underline{M}}*(1./K))$

or simply the general power-low mean:

$$\boxed{\underline{\text{Kmean}} = (\underline{\underline{M}} * \underline{K}.\hat{}\,p).\hat{}\,(1/p)} \qquad p \neq 0$$

Then we place it on diagonal

Kd = spdiags($\underline{\text{Kmean}}$, 0, Nf, Nf)