Generating Random Permeability Fields

Subsurface is very heterogeneous at all scales!

Some heterogeneity, such as layering has a natural/obvious large scale order.
Other heterogeneity is largely random in nature. Both are important, but the random component introduces uncertainty even if the large scale structure is known.

⇒ useful to be able to generate a large number of distinct random fields with same statistics.

Field of "Geostatistics" or spatial statistics (PGE 337)
⇒ generate lots of software packages that take for ever to install and have 100's of pages of documentation.

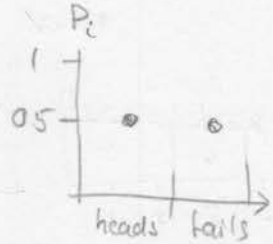
Here we just need the most basic functionality.

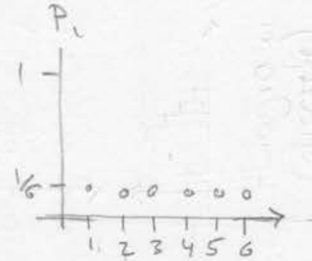
Generate random Gaussian fields with a specified covariance structure and mean.

Some language

Random variable X:

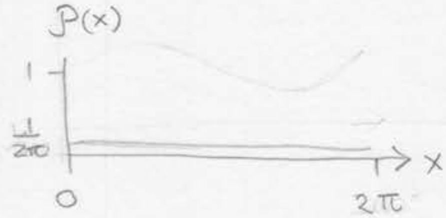
- discrete random variable
  coin toss, dice roll
  finite number of outcomes
  $\Rightarrow$ discrete probability p_i





- continuous random variable:

"spinner" can land on any location $\rightarrow$ continuous



$\Rightarrow$ continuous probability density function $p(x)$

Parameters describing GW aquifers are continuous.

Expected value (mean):

discrete case: $E(X) = \sum_{i=1}^{k} x_i p_i = \mu$ $\quad x_i$ i-th outcome, p_i i-th probability

continuous case $\quad E(X) = \int_{\mathbb{R}} x\, p(x)\, dx = \mu$

Variance

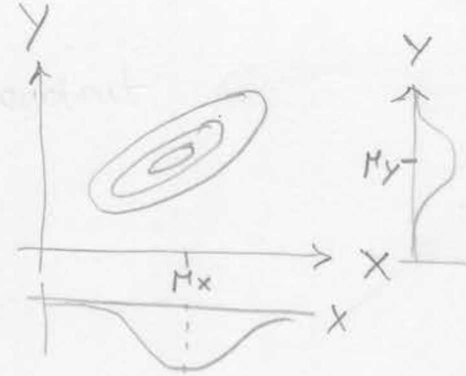
squared deviation from mean

$Var(X) = E[(X-\mu)^2] = E[X^2] - E[X]^2 = \sigma^2(X)$ std. deviation

Discrete case. $Var(X) = \sum_{i=1}^{n} p_i (x_i - \mu)$

Continuous case: $Var(X) = \int_{\mathbb{R}} (x-\mu)^2 p(x)\, dx$

Covariance



If two random variables X and Y are not independent, i.e., they are jointly distributed (p_{ij}, $p(x,y)$) we can compute their covariance

$$COV(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

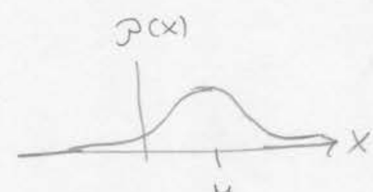
Discrete $\quad COV(X,Y) = \sum_i \sum_j (x_i-\mu_x)(y_j-\mu_y)\, p_{ij}$

Continuous $\quad COV(X,Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y_j-\mu_y)\, p(x,y)\, dx\, dy$

Notes: $\quad COV(X,X) = VAR(X) = \sigma^2(X)$

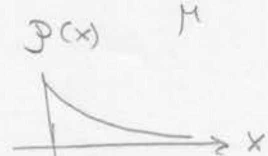
Correlation: $\rho_{x,y} = \dfrac{COV(X,Y)}{\sigma_x \cdot \sigma_y}$ $\quad$ scaled covariance

Typical probability density distributions



Normal distribution: $\quad p(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

$$E(X) = \mu \quad Var(X) = \sigma^2$$



Exponential distribution: $\quad p(x) = \lambda e^{-\lambda x}$

$$E(X) = \lambda^{-1} \quad Var(X) = \lambda^{-2}$$

Random functions/fields/processes

So far no spatial/temporal extent !

Random field $Z = \{ Z(\underline{x}) : \underline{x} \in \mathbb{R}^d \}$

where $Z(\underline{x})$ is a scaler random variable ~~at x~~
at location $\underline{x}$.

If the field/process is stationary, i.e, statistics do not depend on $\underline{x}$ then we can define covariance

$$C(h) = \mathrm{Cov}(Z(\underline{x}), Z(\underline{x}+\underline{h})) \qquad \underline{h} \in \mathbb{R}^d$$

where $\underline{h}$ is the lag/distance vector $\underline{h} = \underline{x} - \underline{y}$

For an isotropic random field the covariance is only a function of distance $h = |\underline{h}|$

Correlation function $\rho(h) = \dfrac{C(h)}{\sigma^2}$

Common isotropic correlation functions:

1) Power exponential: $\rho(h) = \exp\left(-\left(\frac{h}{\kappa}\right)^\nu\right)$, $\kappa > 0$, $0 < \nu \leq 2$
$\nu = 2 \rightarrow$ gaussian

2) Rational quadratic (Cauchy) $\rho(h) = \dfrac{1}{\left(1+\left(\frac{h}{\kappa}\right)^2\right)^\nu}$ $\kappa > 0, \nu > 0$

3) Matérn. $\rho(h) = \dfrac{1}{\Gamma(\nu) 2^{\nu-1}} \left(\dfrac{2\sqrt{\nu} h}{\kappa}\right)^\nu K_\nu\left(\dfrac{2\sqrt{\nu} h}{\kappa}\right)$, $\kappa > 0$, $\nu > 0$

K_ν mod. Bessel function

Γ Gamma function

Matérn covariance fields

Parameterized family of covariance fields that contains most commonly used functions as limiting cases.

$\nu = \frac{1}{2}$ → exponential correlation function

$\nu \to \infty$ → Gaussian correlation function

κ controlls how fast the correlation decays with distance.

Relation to Stochastic PDE's

The real benefit of Matérn covariances is that they have been linked explicitly to SPDE's.

In particular to following PDE:

$$(-\nabla^2 + \kappa^2)^{\alpha/2}\, m = s$$

where s = white noise Gaussian random field with unit marginal variance

$\alpha = \nu + \frac{d}{2}$

d = dimension

m = unknown parameter

was only shown in 2011 by Lindgren

already cited 634 times!

Why is this useful?

We have to solve PDE, but we do not have to form the Covariance matrix $\underline{\underline{C}}$ explicity.

$$C_{ij} = \mathrm{Cov}(\underline{x}_i, \underline{x}_j)$$

Standard sampling of field with specified covariance.

$$\underline{m} = \underline{\underline{L}}\,\underline{s} + \mu$$

μ = mean

$\underline{s}$ = white noise

L = Cholesky decomp. of $\underline{\underline{C}}$ so that $\underline{\underline{C}} = \underline{\underline{L}}^T \underline{\underline{L}}$

Requires that we compute factorization $\underline{\underline{L}}$ of $\underline{\underline{C}}$ which is dense! (Limits size of field we can compute)

In contrast, discrete operators for SPDE are sparse and can be solved efficiently

Our Matérn parameterization

$$(-\gamma \nabla^2 + \delta)^{\alpha/2} m = s$$

two parameters γ, δ rather than one, κ.
(difference must be that κ version assumes $\gamma=1$)

We choose $\alpha = 2$, so that we have simple PDE.

SPDE: $(-\gamma \nabla^2 + \delta) m = s$ on Ω

BC: $-\gamma \nabla m = 0$ on $\partial\Omega$ (natural BC's)

We can show correlation length

$\rho = 2\sqrt{\gamma/\delta}$ $\sigma^2 \sim \frac{1}{\gamma\delta}$