Lecture 11: Intro to 2D numerics

Logistics: - HW1 last chance Thursday !

⇒ come to office hrs

- HW3 due Thursday

⇒ get started

Last time: - Neumann / Flux BC

$\Rightarrow \quad f_n = q_b \frac{A}{V}$ equivalent source term

$\underline{\underline{L}}\,\underline{h} = \underline{f}_s + \underline{f}_N$

- Compute fluxes

$\underline{q} = -\underline{\underline{Kd}}\;\underline{\underline{G}}\;\underline{h}$ → interior

$\underline{q} = \pm\, \underline{r}\, \frac{V}{A}$ → bnd

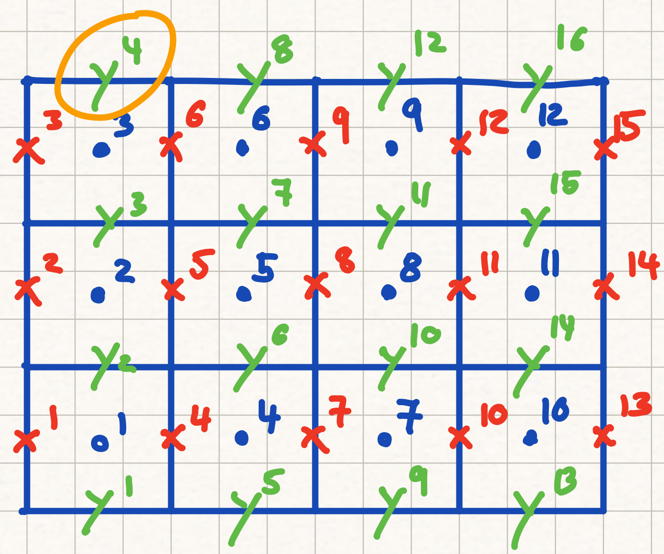
$\underline{r} = \underline{\underline{L}}\,\underline{h} - \underline{f}_s$ residual

Today: Transition to 2D

- Matlab basics
- 2D staggered grid
- Discrete gradient & divergence

Staggered grid in 2D

$N_x = 4 \qquad N_y = 3 \qquad N = N_x \cdot N_y = 12$



Number y-first

faces in x-dir:

$N_{fx} = (N_x + 1) \cdot N_y$

faces in y-dir:

$N_{fy} = N_x \cdot (N_y + 1) = 16$

Total faces : $N_f = N_{fx} + N_{fy} = 31$

Discrete Gradient in 2D

Continuous gradient: $\nabla h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix}$

approx. $\frac{\partial h}{\partial x} \sim \underline{dhx}$ on x-faces

approx. $\frac{\partial h}{\partial y} \sim \underline{dhy}$ on y-faces

Choose to build $\underline{\underline{G}}$ such that

$$\underline{dh} = \begin{bmatrix} \underline{dhx} \\ \underline{dhy} \end{bmatrix}$$

⇒ 2D gradient can be decomposed as

$$\underline{\underline{G}} = \begin{bmatrix} \underline{\underline{G_x}} \\ \underline{\underline{G_y}} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \underline{dhx} &= \underline{\underline{G_x}}\, \underline{h} \\ \underline{dhy} &= \underline{\underline{G_y}}\, \underline{h} \end{aligned}$$

$$\begin{array}{c} \\ N_{fx} \\ \\ N_{fy} \end{array} \overset{\underline{dh}}{\begin{bmatrix} \underline{dhx} \\ \\ \underline{dhy} \end{bmatrix}} = \overset{\underline{\underline{G}}}{\overbrace{\begin{bmatrix} \underline{\underline{G_x}} \\ \\ \underline{\underline{G_y}} \end{bmatrix}}^{N}} \overset{\underline{h}}{\begin{bmatrix} \; \end{bmatrix}} \, N$$

Matrix dimension:

$\underline{\underline{G}}$ is $N_f \cdot N$

$\underline{\underline{G_x}}$ is $N_{fx} \cdot N$

$\underline{\underline{G_y}}$ is $N_{fy} \cdot N$

Discrete divergence

dy

$$f_s = \nabla \cdot q = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \approx \underline{\underline{D}}\, q = \underline{\underline{D_x}}\, q_x + \underline{\underline{D_y}}\, q_y$$

$$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = -\underline{Kd}\, \underbrace{\underline{\underline{G}}\, \underline{h}}_{\begin{bmatrix} dhx \\ dhy \end{bmatrix}}$$

$$\begin{array}{cccc} & f_s & Nf_x \qquad Nf_y & q \\ N & \begin{bmatrix} \\ \\ \end{bmatrix} & = N \left[\, \underline{\underline{D_x}} \;\middle|\; \underline{\underline{D_y}} \,\right] & \begin{bmatrix} q_x \\ q_y \end{bmatrix} \begin{matrix} \}\, Nf_x \\ \}\, Nf_y \end{matrix} \\ & 1 & & \end{array}$$

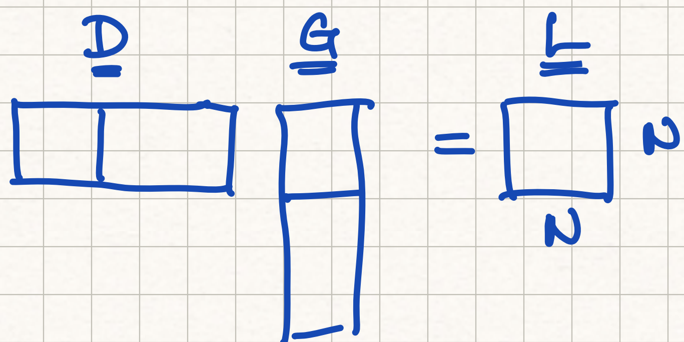
Matrix dimensions

$\underline{\underline{D}}$ is $N \cdot Nf$

$\underline{\underline{D_x}}$ is $N \cdot Nf_x$

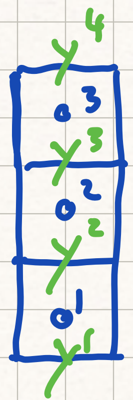
$\underline{\underline{D_y}}$ is $N \cdot Nf_y$

$$\underset{N \cdot Nf}{\underline{\underline{D}}} * \underset{Nf \cdot N}{\underline{\underline{G}}}$$



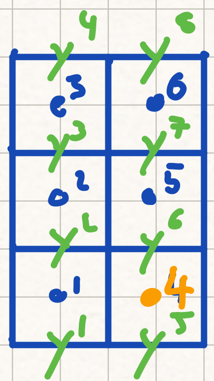
Building 2D Divergence matrix

Start with $\underline{D_y}$ in 1D $\qquad \nabla \cdot q = f_s$



$$\underset{f_s}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \frac{1}{\Delta y} \underset{N_y \cdot (N_y+1)}{\underbrace{\overset{\underline{D_y}^1}{\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix}}}} \underset{q_y}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}$$

Suppose we add second column

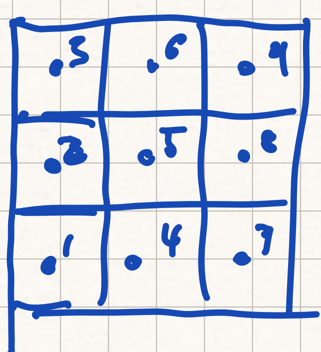


$$\underset{f_s}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}} = \frac{1}{\Delta y} \overset{\underline{D_y}^2}{\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}} \begin{bmatrix} -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & -1 & 1 & & & & \\ & & & & -1 & 1 & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \end{bmatrix} \underset{q_y}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}}$$

N_y $\qquad N_y+1$

$$\underline{\underline{D_y}}^2 = \begin{bmatrix} \underline{\underline{D_y}}^1 & \\ & \underline{\underline{D_y}}^1 \end{bmatrix}$$

2 x 2 block matrix with $\underline{\underline{D_y}}^1$ on diagonal block



$$\underline{\underline{D_y}}^2 = \begin{bmatrix} \underline{\underline{D_y}}^1 & & \\ & \underline{\underline{D_y}}^1 & \\ & & \underline{\underline{D_y}}^1 \end{bmatrix}$$

In general:

$\underline{\underline{Dy}}^2$ is a block matrix with Nx by Nx block of size Ny by (Ny+1). Diagonal blocks are $\underline{\underline{Dy}}^1$ all others are zero.

Tensor product construction of $\underline{\underline{Dy}}^2$

The discrete 2D operator can easily and efficiently be assembled using Kronecker/Tensor product.

Definition:

If $\underline{\underline{A}}$ is a $m \times n$ matrix and $\underline{\underline{B}}$ is a $p \times q$ matrix then the Kronecker product $\underline{\underline{A}} \otimes \underline{\underline{B}}$ is $mp \times nq$ block matrix

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & & \\ & & a_{mn} \end{bmatrix} \qquad B$$

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{bmatrix} a_{11}\underline{\underline{B}} & \cdots & a_{1n}\underline{\underline{B}} \\ \vdots & \ddots & \\ a_{m1}\underline{\underline{B}} & & a_{mn}\underline{\underline{B}} \end{bmatrix}$$

How can we use this to assemble $\underline{\underline{Dy}}$

$$\underline{\underline{Dy}}^2 = \underline{\underline{Ix}} \otimes \underline{\underline{Dy}}' = \begin{bmatrix} \underline{\underline{Dy}}' & & & & \\ & \underline{\underline{Dy}}' & & & \\ & & \underline{\underline{Dy}}' & & \\ & & & \underline{\underline{Dy}}' & \\ & & & & \underline{\underline{Dy}}' \end{bmatrix}$$

$\underline{\underline{Ix}}$ is Nx by Nx identity matrix

In Matlab the tensor product is obtained as:

$$\boxed{\underline{\underline{Dy}} = \mathrm{kron}(\underline{\underline{Ix}}, \underline{\underline{Dy}})}$$

2D op 1D op