Lecture 12: 2D discrete operators

Logistics: - HW 3 is due (Q1: 16 Q2: 15 Q3: 16 Q4: 15 Q5: 15)

No for loops!

- HW1 complete ✓
- HW2 next Thursday last chance to submit
- HW4 will be posted due next Thursday

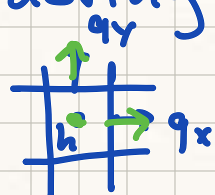
⇒ Remind me if you have extension!

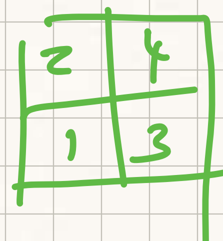
Last time: • Started 2D discretization

- Matlab functions: meshgrid & reshape

  ⇒ y-first ordering

- 2D staggered grid





- $\underline{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ $\underline{dh} = \begin{bmatrix} dh_x \\ dh_y \end{bmatrix}$
- $\Rightarrow \underline{\underline{D}} = [\underline{\underline{D_x}} \ \underline{\underline{D_y}}]$ $\underline{\underline{G}} = \begin{bmatrix} \underline{\underline{G_x}} \\ \underline{\underline{G_y}} \end{bmatrix}$
- 4 matrices we need to build
- $\underline{\underline{D_y}}^2 = \begin{bmatrix} \underline{\underline{D_y}}^1 & & & \\ & \underline{\underline{D_y}}^1 & & \\ & & \underline{\underline{D_y}}^1 & \\ & & & \ddots \end{bmatrix} = \underline{\underline{I_x}} \otimes \underline{\underline{D_y}}$ kron

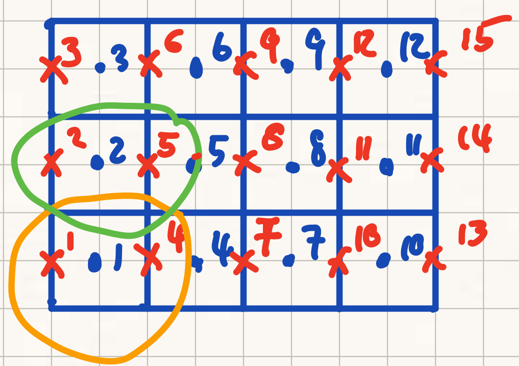
Today: - $\underline{\underline{D_x}}$, $\underline{\underline{G_x}}$, $\underline{\underline{G_y}}$, $\underline{\underline{M_x}}$, $\underline{\underline{M_y}}$
- Testing and Convergence
- Code transition from 1D to 2D

Build $\underline{\underline{Dx}}$

On a x-first $\underline{\underline{Dx}}^2 = \underline{\underline{I_y}} \otimes \underline{\underline{Dx}}^1$

What does $\underline{\underline{Dx}}$ look like on y-first grid?

$N_x = 4 \qquad N_y = 3$



$$f_{s_1} = \frac{q_4 - q_1}{\Delta x}$$

$\underline{\underline{Dx}}^2$

$$\underline{fs} = \frac{1}{\Delta x}\;\begin{array}{c|ccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline
1 & -1 & & & 1 & & & & & & & & & & & \\
2 & & -1 & & & 1 & & & & & & & & & & \\
3 & & & -1 & & & 1 & & & & & & & & & \\
4 & & & & -1 & & & 1 & & & & & & & & \\
5 & & & & & -1 & & & 1 & & & & & & & \\
6 & & & & & & -1 & & & 1 & & & & & & \\
7 & & & & & & & -1 & & & 1 & & & & & \\
8 & & & & & & & & -1 & & & 1 & & & & \\
9 & & & & & & & & & -1 & & & 1 & & & \\
10 & & & & & & & & & & -1 & & & 1 & & \\
11 & & & & & & & & & & & -1 & & & 1 & \\
12 & & & & & & & & & & & & -1 & & & 1
\end{array}\;\underline{q_x}$$

$\Rightarrow \underline{\underline{Dx}}^2$ is sparse diagonal matrix

(could be assembled with spdiags)

$\underline{\underline{Dx}}^2$ is also a block matrix

that can be assembled by tensor/Kronecker

product $\underline{\underline{A}} \otimes \underline{\underline{B}}$!

$\underline{\underline{B}} = \underline{\underline{I_y}}$ $\qquad \underline{\underline{I_y}}$ is $N_y \cdot N_y$ Identity

$$\underline{\underline{Dx}} = \frac{1}{\Delta x} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\underline{\underline{I_y}} & \underline{\underline{I_y}} & & & & \\ & -\underline{\underline{I_y}} & \underline{\underline{I_y}} & & & \\ & & -\underline{\underline{I_y}} & \underline{\underline{I_y}} & & \\ & & & -\underline{\underline{I_y}} & \underline{\underline{I_y}} & \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} = \Delta x \, \underline{\underline{Dx}}'$$

$$\underline{\underline{Dx}}^2 = \underline{\underline{Dx}}' \otimes \underline{\underline{I_y}}$$

Kronecker product assembly of $\underline{\underline{D}}$

$$\boxed{\begin{aligned} \underline{\underline{Dx}} &= \underline{\underline{Dx}} \otimes \underline{\underline{I_y}} \\ \underline{\underline{Dy}} &= \underline{\underline{I_x}} \otimes \underline{\underline{Dy}} \end{aligned}}$$

$$\underline{\underline{D}} = [\underline{\underline{Dx}} \ \ \underline{\underline{Dy}}]$$

Discrete Gradient in 2D

The $\underline{\underline{G_x}}$ and $\underline{\underline{G_y}}$ matrices can also be built with Kronecker products

$$\underline{\underline{G_x}} = \underline{\underline{G_x}} \otimes \underline{\underline{I_y}}$$
$$\underline{\underline{G_y}} = \underline{\underline{I_x}} \otimes \underline{\underline{G_y}}$$

But we can also the adjoint relation between $\underline{\underline{D}}$ and $\underline{\underline{G}}$

$$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T}$$ true in interior

Need to impose natural BC's

$\Rightarrow$ set $\underline{\underline{G}}$ to zero on Bnds

```
dof_f_bnd = [dof_f_xmin; dof_f_xmax; ...
             dof_f_ymin; dof_f_ymax];
```

Zero rows out:

```
G(dof_f_bnd, :) = 0;
```

2D Mean Operator

$\underline{\underline{M}}$ have same structure as $\underline{\underline{G}}$

$$\underline{\underline{G}} = \begin{bmatrix} \underline{\underline{G_x}} \\ \underline{\underline{G_y}} \end{bmatrix} \Rightarrow \underline{\underline{M}} = \begin{bmatrix} \underline{\underline{M_x}} \\ \underline{\underline{M_y}} \end{bmatrix}$$

Just like $\underline{\underline{G}}$ we can assemble $\underline{\underline{M_x}}$ and $\underline{\underline{M_y}}$ from respective 1D operators

$$\boxed{\begin{aligned} \underline{\underline{M_x}} &= \underline{\underline{M_x}} \otimes \underline{\underline{I_y}} \\ \underline{\underline{M_y}} &= \underline{\underline{I_x}} \otimes \underline{\underline{M}} \end{aligned}}$$

We won't deal with curl ($\nabla \times$) in this class