Lecture 13: Streamlines & Stream function

Logistics: - Office hours today only 3-3:30pm!
- HW4 due Thursday
- HW2 last chance on Thursday (remember you need to complete entire HW to get credit)

Last time: - 2D discrete operators
- Assemble from 1D ops with Kronecker products: $\underline{\underline{A}} \otimes \underline{\underline{B}}$

2D

$$\underline{\underline{D_x}} = \underline{\underline{D_x}} \otimes \underline{\underline{I_y}}$$
$$\underline{\underline{D_y}} = \underline{\underline{I_x}} \otimes \underline{\underline{D_y}}$$
$$\underline{\underline{D}} = [\underline{\underline{D_x}}, \underline{\underline{D_y}}]$$

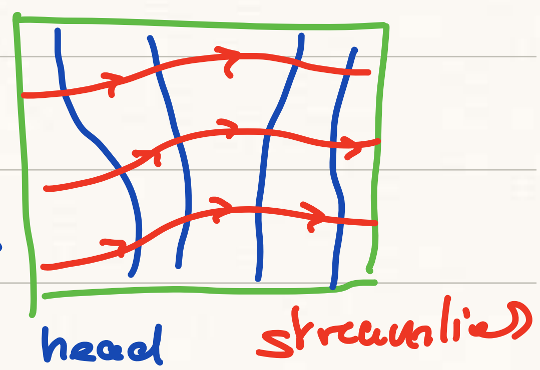
$$\underline{\underline{G_x}} = \underline{\underline{G_x}} \otimes \underline{\underline{I_y}}$$
$$\underline{\underline{G_y}} = \underline{\underline{I_x}} \otimes \underline{\underline{G_y}}$$
$$\underline{\underline{G}} = \begin{bmatrix} \underline{\underline{G_x}} \\ \underline{\underline{G_y}} \end{bmatrix}$$

$\underline{\underline{M}}$ similar to $\underline{\underline{G}}$

Today: - Visualizing 2D Flow Fields

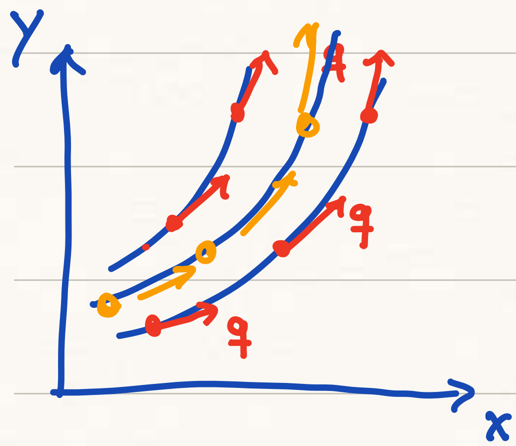
$\Rightarrow$ Flow net

1. Head contours →
2. Streamlines →



Streamlines & Streamfunction

Stream lines provide best way to illustrate flow field, if applicable:



Definition:

Streamlines are the family of curves that are instantaneously tangent to the velocity field.

In a steady flow the stream lines are particle path!

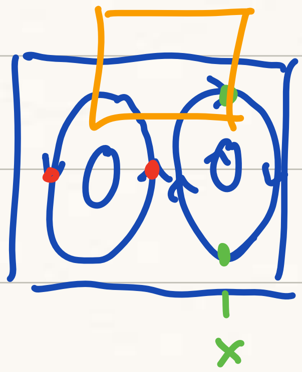
Porous media : $q = \phi \underline{v}$ $\quad q \parallel \underline{v}$

How do we compute Streamlines?

The definition of velocity provides a system of ODE's to compute streamlines: $\underline{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

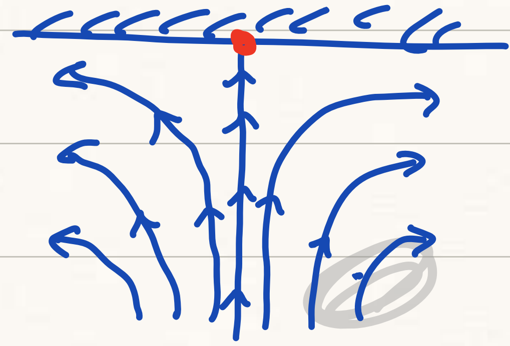
$$\left.\begin{array}{l} 1) \quad \frac{dx}{dt} = v_x(\underline{x}) \\ 2) \quad \frac{dy}{dt} = v_y(\underline{x}) \end{array}\right\} \qquad \frac{dy}{dx} = \frac{v_y}{v_x}$$

We can either solve system or single ODE to get one particular streamline through $\underline{x}_0$.



$\frac{dy}{dx} = \infty$

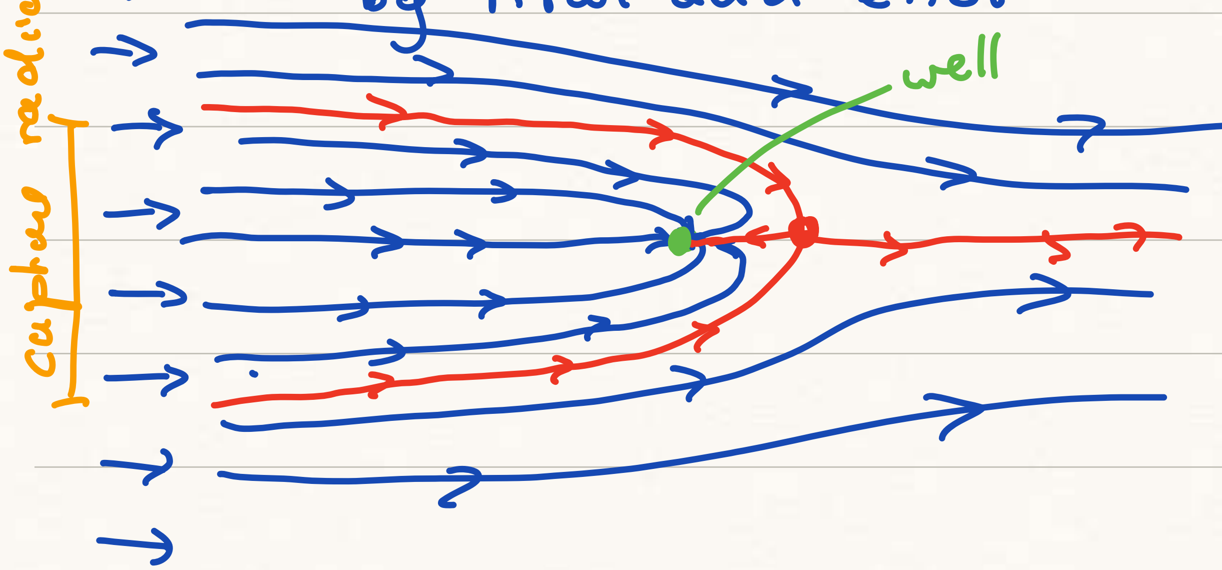
stagnation point



Notes: • Safer to solve system of ODE's because $\frac{dy}{dx}$ may be infinite and y could be multivalued function of x.

• The ODE system has problems with stagnation points, because $\underline{v} \to \underline{0}$

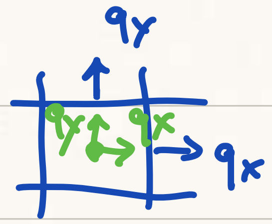
• Can only determine stagnation points by trial and error



Matlab has function streamline.m

to solve for streamline given initial point.

Note: need to interpolate fluxes

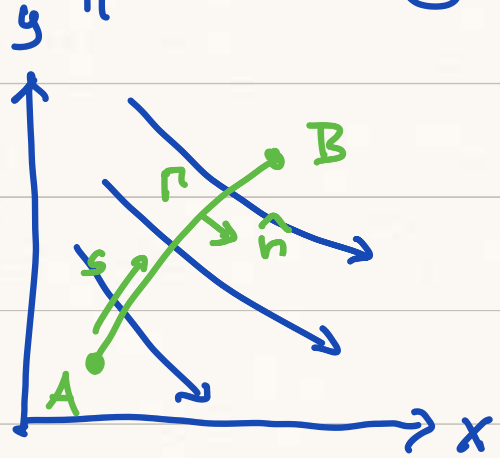


to cell centers to use

in built function

⇒ interpolation can introduce errors

Different way of thinking about streamlines



Cummulative flux between A & B

$$\boxed{\psi = \int_{\Gamma} q \cdot \hat{n}\, ds}$$

Γ = path

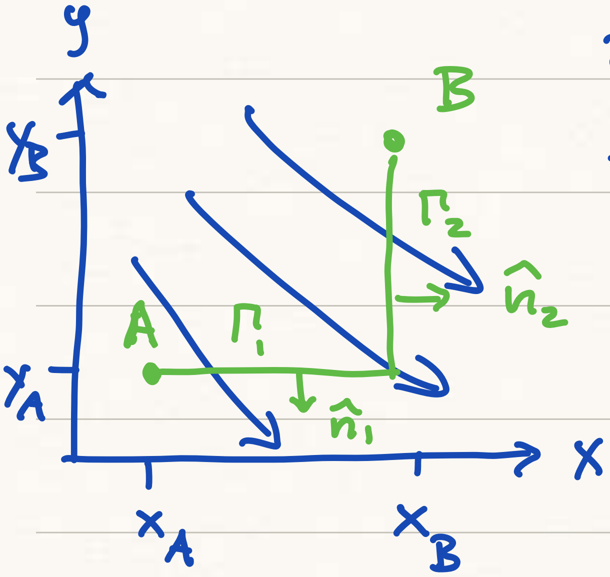
s = arc length

$\hat{n}$ = right hand normal

In the absence of sources/sinks

ψ should not depend on path Γ.

⇒ choose simplest path



along Γ_1: $q \cdot \hat{n}_1 = (q_x \;\; q_y)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -q_y$

along Γ_2: $q \cdot \hat{n}_2 = (q_x \;\; q_y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = q_x$

Rewrite our integral: $\psi = \underbrace{\int_{x_A}^{x_B} -q_y(x, y_A)\, dx}_{\Gamma_1} + \underbrace{\int_{y_A}^{y_B} q_x(x_B, y)\, dy}_{\Gamma_2}$

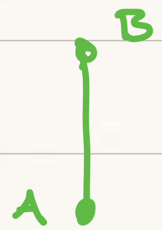
Consider two limits:

$y_A = y_B$

$$\psi = \int_{x_A}^{x_B} -q_y\, dx = \int_{x_A}^{x_B} \frac{\partial \psi}{\partial x}\, dx$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = -q_y$$

$x_A = x_B$:



$$\psi = \int_{y_A}^{y_B} q_x\, dy = \int_{y_A}^{y_B} \frac{\partial \psi}{\partial y}\, dy$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = v_x$$

Therefore:

$$\boxed{\frac{\partial \psi}{\partial x} = -q_y \qquad \frac{\partial \psi}{\partial y} = q_x}$$

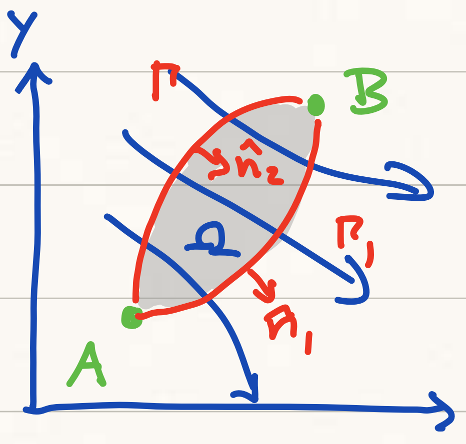
This is often given as the definition of the stream function ψ.

Physical Interpretation:

ψ is cumulative flux

- Change of cumulative flux in x-direction is proportional to negative flux in y-direction.
- Change of cumulative flux in y-direction is proportional to flux in x-direction.

All of this is true if the integral defining ψ is path-independent.



$$\int_{\Gamma_1} q \cdot \hat{n}_1 \, ds = \int_{\Gamma_2} q \cdot \hat{n}_2 \, ds$$

$$\int_{\Gamma_1} q \cdot \hat{n}_1 \, ds - \int_{\Gamma_2} q \cdot \hat{n}_2 \, ds = 0$$

Combine $\Gamma_1 + \Gamma_2 = \Gamma$ and define outside normal: $\hat{n} = \hat{n}_1 \quad \hat{n} = -\hat{n}_2$

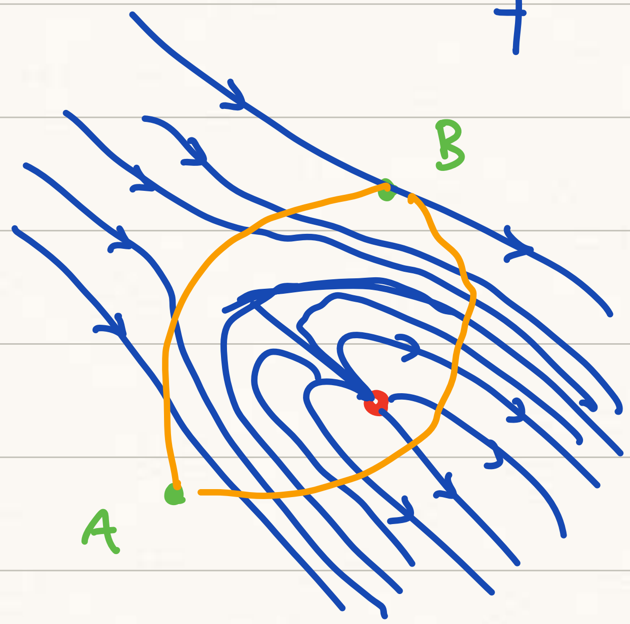
$$\int_{\Gamma_1} q \cdot \hat{n} \, ds + \int_{\Gamma_2} q \cdot \hat{n} \, ds = \oint_{\Gamma} q \cdot \hat{n} \, ds = 0$$

Divergence theorem: $\oint_\Gamma q \cdot n \, ds = \int_\Omega \nabla \cdot q \, dV = 0$

Hence the streamfunction is well defined, i.e., single valued, if $\boxed{\nabla \cdot q = 0}$

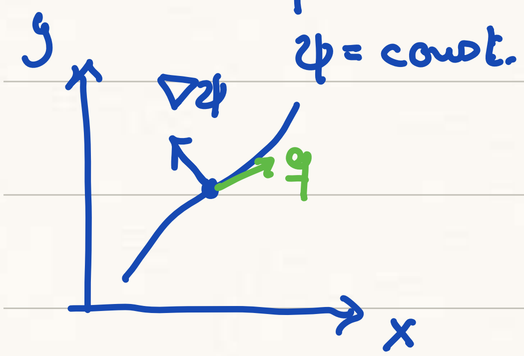
- flow is incompressible
- no sources or sinks

Our eqn: $\left.\begin{aligned} \nabla \cdot q &= f_s \\ q &= -K\nabla h \end{aligned}\right\} -\nabla \cdot K \nabla h = f_s$



Relation between ψ and streamlines

1) The level sets (contours) of ψ are tangential to q $\Rightarrow$ level sets of ψ are streamlines



$$\nabla\psi \cdot q = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) \cdot \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

$$= (-q_y, q_x) \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

$$= -q_y q_x + q_x q_y = 0$$

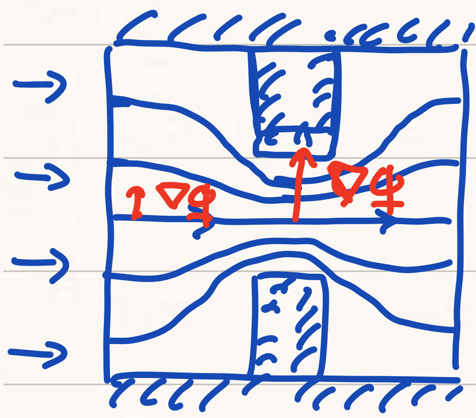
$$\nabla\psi \perp q$$

$\Rightarrow$ ψ contours are parallel to q $\Rightarrow$ streamlines

2) The magnitude of the flux is equal the magnitude of $\nabla\psi$.

$$|\nabla\psi| = \sqrt{\frac{\partial\psi}{\partial x}^2 + \frac{\partial\psi}{\partial y}^2} = \sqrt{(-q_y)^2 + (q_x)^2}$$

$$= \sqrt{q_x^2 + q_y^2} = |q|$$



$k = 0$