Lecture 19: Advection

Logistics:
- HW6 due today
- HW7 is posted due next Thursday
- HW4 last chance this Thursday
- Office hours on Zoom today

Last time:
- Timestepping $\quad \underline{c}^{\theta} = \theta \underline{c}^{n} + (1-\theta)\underline{c}^{n+1}$
- Theta Method

$$\boxed{\underline{\underline{IM}}\, \underline{c}^{n+1} = \underline{\underline{Ex}}\, \underline{c}^{n} + \Delta t\, \underline{f_s}}$$

$$\underline{\underline{IM}} = \underline{\underline{\Phi}}^{s} + \Delta t (1-\theta)\, \underline{\underline{L}} \qquad \theta = 1: \text{ Forward E.}$$

$$\underline{\underline{Ex}} = \underline{\underline{\Phi}} - \Delta t\, \theta\, \underline{\underline{L}} \qquad \theta = 0: \text{ Backw. E}$$

- Same for all linear problems $\quad \theta = \frac{1}{2}$: CN
- Physics contained in $\underline{\underline{L}}$

Today:
- Advection
- Method of Characteristics
- Advection operator

Advection

Solute balance equation

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot [\underbrace{\underset{\cdot}{q} c}_{\text{Advection}} - D_m \nabla c] = -\phi k c$$

nondimensionalize:

$$\frac{\partial c'}{\partial t'} + \nabla' \cdot \left[\frac{t_c}{t_A} q' c' - \frac{t_c}{t_D} \nabla' c'\right] = -\frac{t_c}{t_R} c'$$

last time $t_c = t_D$ take limit $Pe = \frac{t_D}{t_A} \rightarrow 0$

$\Rightarrow$ Diffusion Equ

today: $t_c = t_A$

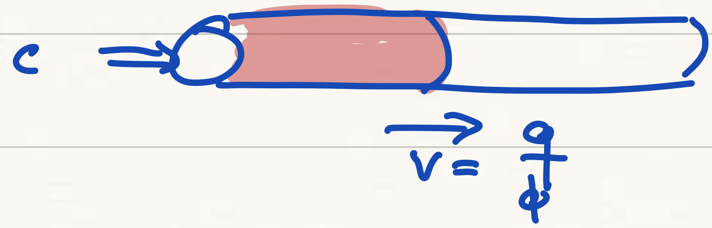
$$\frac{\partial c'}{\partial t'} + \nabla' \cdot \left[q' c' - \frac{1}{\underline{\underline{Pe}}} \nabla' c'\right] = -Da\, c'$$

limits $Pe \rightarrow \infty$ $Da \rightarrow 0$

$\Rightarrow$ Advection equation: $\boxed{\frac{\partial c'}{\partial t'} + \nabla \cdot [q' c'] = 0}$

Analytic solutions to Advection eqn

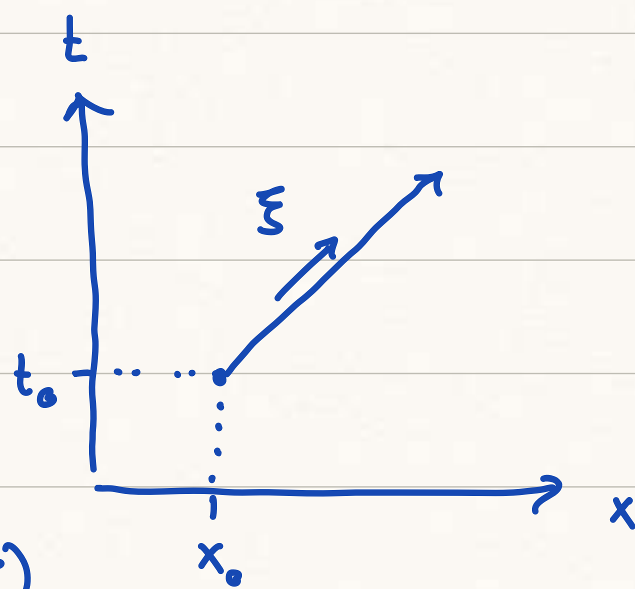
Consider 1D transport of tracer along 1D column with const ϕ, k $\Rightarrow$ q & v = const



Dimensional Problem:

$$\boxed{\begin{array}{ll} \text{PDE:} & \dfrac{\partial c}{\partial t} + v \dfrac{\partial c}{\partial x} = 0 \\ \text{IC} & c(x,0) = c_0(x) \end{array}} \qquad x \in \mathbb{R} \qquad t \in [0, T]$$

Method of characteristics



Idea: Find a characteristic curve, ξ, along which PDE reduces to an ODE

$$c(x,t) = c(x(\xi), t(\xi)) = \theta(\xi)$$

Total change of θ along ξ

$$\frac{d\theta}{d\xi} = \frac{\partial c}{\partial x}\frac{dx}{d\xi} + \frac{\partial c}{\partial t}\frac{dt}{d\xi}$$

rearrange

$$\frac{d\theta}{d\xi} = \frac{dt}{d\xi}\frac{\partial c}{\partial t} + \frac{dx}{d\xi}\frac{\partial c}{\partial x}$$

PDE: $$1\,\frac{\partial c}{\partial t} + v\,\frac{\partial c}{\partial x} = 0$$

1) $\frac{d\theta}{d\xi} = 0$

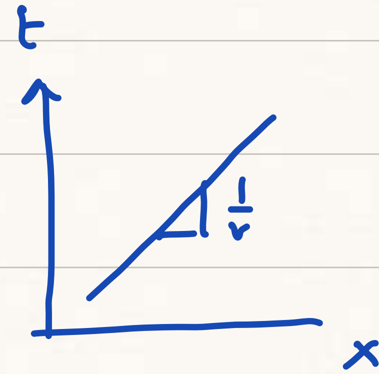
2) $\frac{dt}{d\xi} = 1$

3) $\frac{dx}{d\xi} = v$

$\left.\begin{matrix} 2) \\ 3) \end{matrix}\right\} \frac{dx}{dt} = v$

Solve equ for characteristic curve:

$$x - x_0 = v(t - t_0)$$



Initial condition:

$$c(x = x_0, t = t_0) = c_0(x_0)$$

substitute characteristic $\quad x_0 = x - v(t - t_0)$

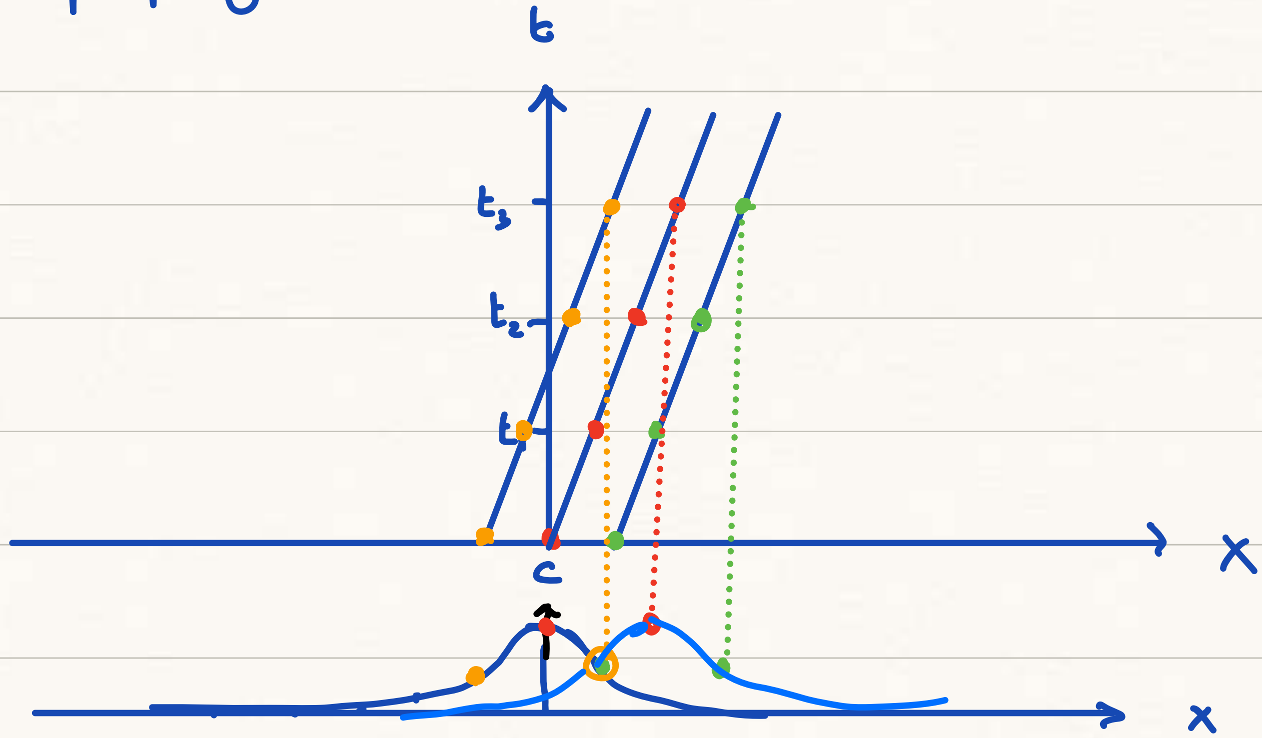
Analytic soln: $\boxed{c(x,t) = c_0(x - v(t - t_0))}$

typically $t_0 = 0$

$$c(x,t) = c_0(\underbrace{x - vt}_{\text{travelling wave coord.}})$$

$\Rightarrow$ Soln. is a Kinematic wave

A wave is a signal/disturbance/variation moving through a medium at a recognizable speed of propagation.



$t_0 = 0$

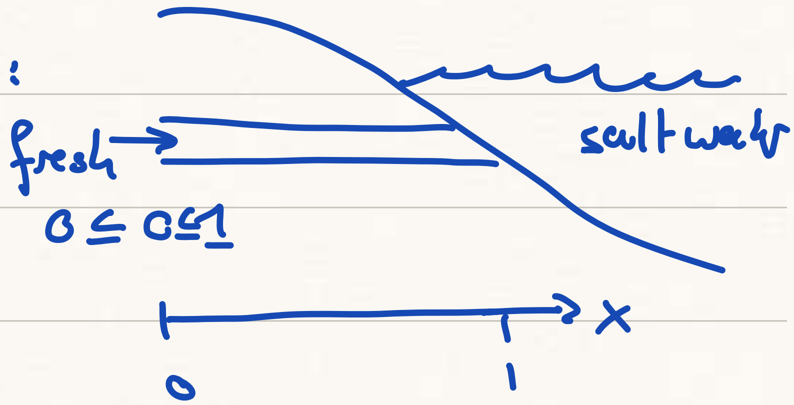
$x_0 = 0$

Discretization of advective flux

Steady example problem

Dimless after dropping primes:

PDE: $\nabla \cdot (Pe\, c - \nabla c) = 0$



BC: $c(0) = 0 \qquad c(1) = 1$

Analytic solution: $\boxed{c(x) = \dfrac{e^{Pe\,x} - 1}{e^{Pe} - 1}}$

Discretize Steady Advection-Diffusion Equ

PDE: $\nabla \cdot (q c - D_m \nabla c) = 0$ q's are known from the flow problem

Discrete: $\underline{\underline{D}} \cdot (\underline{\underline{A}}(q)\underline{c} - \underline{\underline{Kd}}\,\underline{\underline{G}}\,\underline{c}) = \underline{0}$

$$\underbrace{\underline{\underline{D}} \cdot (\underline{\underline{A}}(q) - \underline{\underline{Kd}}\,\underline{\underline{G}})}_{\underline{\underline{L}}}\, \underline{c} = \underline{0}$$

$\underline{\underline{L}}\,\underline{c} = \underline{0}$ as before

What is $\underline{\underline{A}}$?

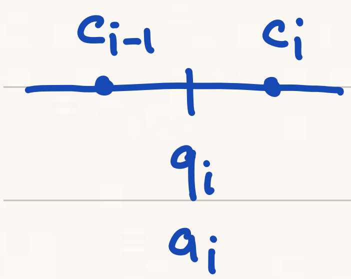
Shape is 1D $Nx+1$ by Nx } same shape as $\underline{\underline{G}}$

2D Nf by N

$\underline{\underline{A}}(q)\,\underline{c} \approx q\,\underline{c}$

Problem: q is on cell faces

$\underline{c}$ is ~~on~~ in cell centers



Advective flux on i-th face

$$\boxed{a_i = q_i \, c_{i-\frac{1}{2}}}$$

How do we approximate $c_{i-\frac{1}{2}}$

<u>Central flux</u> : $c_{i-\frac{1}{2}} = \frac{1}{2}(c_{i-1} + c_i)$

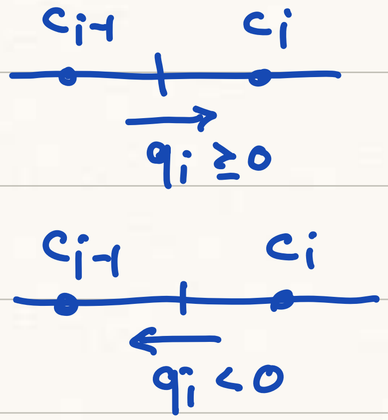
$\underline{\text{c\_cent}} = \underline{\underline{M}}\,\underline{c}$

$\underline{A} = \underline{\underline{Qd}}\,\underline{\underline{M}}$

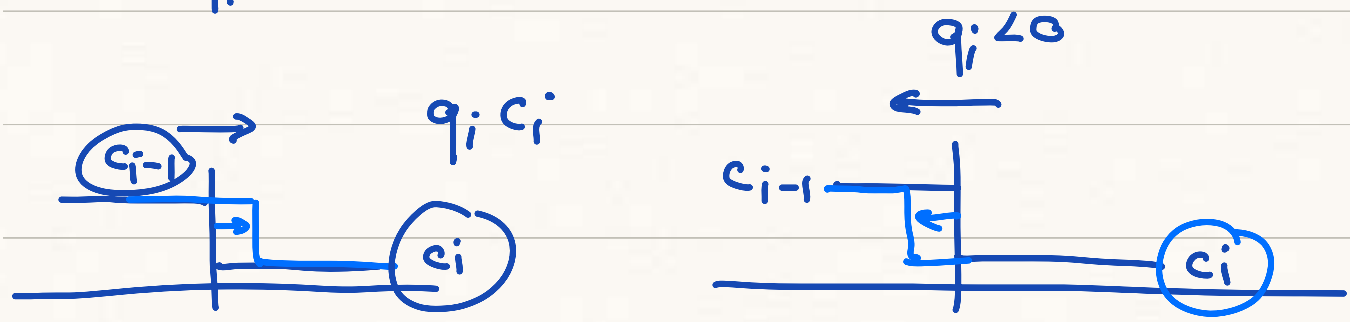
diagonal matrix with q on diagonal

$\Rightarrow$ causes oscillations

Upwind flux



$$\boxed{\begin{aligned} a_i &= q_i \; c_{i-\frac{1}{2}} \\ c_{i-\frac{1}{2}} &= \begin{cases} c_{i-1}, & q_i \geq 0 \\ c_i, & q_i < 0 \end{cases} \end{aligned}}$$



How do we implement this ?