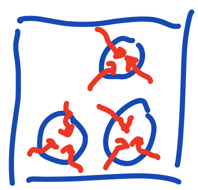
Lecture 2: Conservation Laws

Logistics: Office hours → Tu 3–4 pm

- most fundamental eqns in science come from Balance laws
- Balance laws account for gains & losses of a quantity due to transport and sources/sinks.
- if there are no sources/sinks ⇒ conserved quantity
- In multiphase systems conserved quantities are not always obvious.



  - mass of fluid is conserved
  - energy in fluid is not necessarily conserved. because energy transfer from fluid ⇄ grains

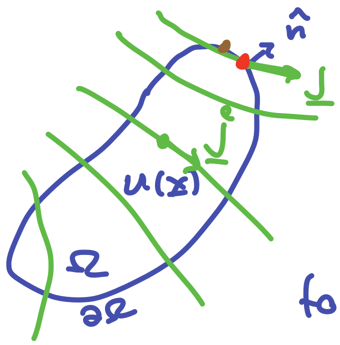
Note on units:

L = unit of length $\#$ = arbitrary unit

T = unit of time 1 = dimensionless

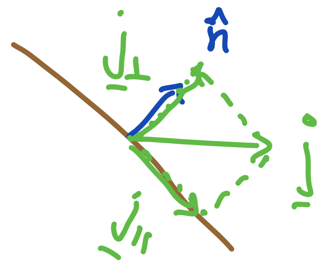
M = unit of mass ϕ $[1]$

Derivation of general balance law



account for changes of unknown $u(\underline{x},t)$ in domain Ω due to fluxes, $\underline{j}$, accross boundary and due to production/consumption in Ω, $\hat{f}_s$.

- $u(x,t)$ is a density $\left[\frac{\#}{L^3}\right]$
- $\underline{j}(x,t)$ is a flux $\left[\frac{\#}{L^2 T}\right]$
- $\hat{f}_s(x,t)$ is volumetric rate $\left[\frac{\#}{L^3 T}\right]$
  - source/sink



General integral balance on Ω:

$$\boxed{\frac{d}{dt} U = J + F}$$

1) U is total amount of u in Ω : $U(t) = \int_\Omega \underbrace{u(x,t)\, dV}_{\frac{\#}{\cancel{L^3}}\ \cancel{L^3}}$

units $[\#]$

2) J is total rate of transport of $\underline{u}$ across $\partial\Omega$ by $\underline{j}$: $J(t) = -\oint_{\partial\Omega} \underbrace{\underline{j}\cdot\hat{n}\, dA}_{\frac{\#}{\cancel{L^2}T}\ 1\ \cancel{L^2}}$

$\hat{n}$ is outward unit normal

ζ $\left[\frac{\#}{T}\right]$ rate

3) F is total rate of prod./consump. of u in Ω

$$F(t) = \int_{\Omega} \hat{f}_s \, dV$$

$$\frac{\#}{\cancel{L^3} T} \quad \cancel{L^3}$$

units of F $\left[\frac{\#}{T}\right]$

Substitute into general balance:

$$\boxed{\frac{d}{dt}\int_{\Omega} u \, dV = -\oint_{\partial\Omega} \underline{j}\cdot\hat{n} \, dA + \int_{\Omega} \hat{f}_s \, dV}$$

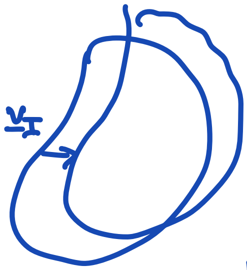
Integral balance law

$$\frac{1}{T} \quad \# \qquad\qquad \frac{\#}{T} \qquad\qquad \frac{\#}{T}$$

To obtain a PDE we need to:

1) Exchange derivative & integral
2) Transform surface integral to volume integral
3) Localization.

1) Reynolds Transport Theorem



If domain Ω is moving with velocity $\underline{v}_I$

$$\frac{d}{dt}\int_{\Omega(t)} u(\underline{x},t)\,dV = \int_{\Omega} \frac{\partial u}{\partial t}\,dV + \oint_{\partial\Omega} u\,\underline{v}_I\cdot\hat{n}\,dS$$

In Eulerian limit of fixed domain

$$\underline{v}_I = 0 \quad\Rightarrow\quad \frac{d}{dt}\int_{\Omega} u\,dV = \int_{\Omega}\frac{\partial u}{\partial t}\,dV$$

2) Divergence thm: $\oint_{\partial\Omega} \underline{j}\cdot\underline{\hat{n}}\,dA = \int_{\Omega} \nabla\cdot\underline{j}\,dV$

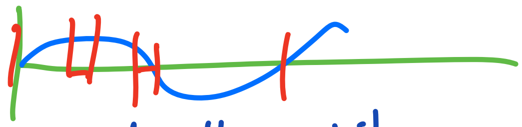
$$\nabla\cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Substitute into integral balance law:

$$\frac{d}{dt}\int u\,dV = -\oint_{\partial\Omega} \underline{j}\cdot\hat{n}\,dV + \int_{\Omega} \hat{f}_s\,dV$$

$$\int_{\Omega}\underbrace{\left(\frac{\partial u}{\partial t} + \nabla\cdot\underline{j} - \hat{f}_s\right)}_{g(\underline{x},t)}dV = 0$$

3) localization $\quad \int_{\Omega} g(\underline{x},t)\,dV = 0$



due to the arbitraryness of Ω the integrand must be zero everywhere: $g = 0$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = \hat{f}_s}$$

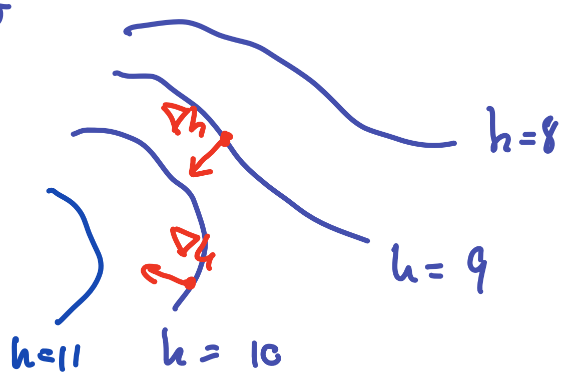
Local form of general balance law

Gradient vs Divergence

Gradient:

Darcy's law: $\underset{\text{vector}}{q} = -\underset{\text{scalar}}{K}\, \nabla \underset{\text{scalar}}{h}$

$$\nabla h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{pmatrix}$$



$h(x,t)$

Remember: 1) no dot

2) takes a scalar gives a vector

3) points "uphill"

Divergence: $\nabla \cdot \underset{\sim}{q} = \text{scalar}$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = \text{scalar}$$

Remember: 1) had dot

2) takes vector and gives scalar

Fluid Mass Balance

General law: $\frac{\partial u}{\partial t} + \nabla \cdot \underset{\sim}{j} = \hat{f}_s$

1) $u = \phi \rho$ — ϕ = porosity $[1]$

ρ = pore fluid density $\left[\frac{M}{L^3}\right]$

$u \Rightarrow \left[\frac{M}{L^3}\right]$

note: assume medium is saturated

2) $\underset{\sim}{j} = \underset{\sim}{v} u = \underset{\sim}{v} \phi \rho$ — $\underset{\sim}{v}$ is ave. interstial velocity

$= \underset{\sim}{q} \rho$ — $\underset{\sim}{v} = \frac{\underset{\sim}{q}}{\phi} \Rightarrow \underset{\sim}{q} = \phi \underset{\sim}{v}$

3) $\hat{f}_s = \rho f_s$ — $f_s \left[\frac{M}{MT} = \frac{1}{T}\right]$

Substitute into general bal. law:

Fluid mass balance: $\boxed{\frac{\partial}{\partial t}(\phi\rho) + \nabla\cdot(q\rho) = \rho f_s}$

Substitue constitutive laws:

1) Darcy's law: $q = -K\nabla h$

2) Equation of state: $\rho = \rho(\underset{\substack{\uparrow\\ \text{pressure}}}{p}) = \rho(h)$ $\qquad h = h(p)$

substitute

$$\boxed{\frac{\partial}{\partial t}(\phi\rho(h)) - \nabla\cdot(K\nabla h) = \rho f_s}$$

$h(x,t)$ is unknown