Lecture 21: Advection operator in 2D

Logistics:
- HW7 is due Thursday
- HW5 last chance Thursday
- Office hours – on zoom

Last time:
- Upwind method $\quad c_{i-\frac{1}{2}} = \begin{cases} c_{i-1} & q \geq 0 \\ c_i & q < 0 \end{cases}$

- Advection Operator 1D

$$\underline{\underline{A}}(q) = \begin{array}{c} \scriptstyle 0 \\ \scriptstyle -1 \end{array}\begin{bmatrix} \ddots & & & \\ \ddots & \ddots & & \\ & qp & qn & \\ & & \ddots & \ddots \end{bmatrix}$$

N_{fx} by N_x

like $\underline{\underline{G}}$

- qn $\quad qn_i = \begin{cases} q_i & q_i < 0 \\ 0 & q_i \geq 0 \end{cases}$

- qp $\quad qp_i = \begin{cases} 0 & q_i \leq 0 \\ q_i & q_i > 0 \end{cases}$

- CFL condition $\quad \boxed{\Delta t \leq \frac{\Delta x}{v}}$

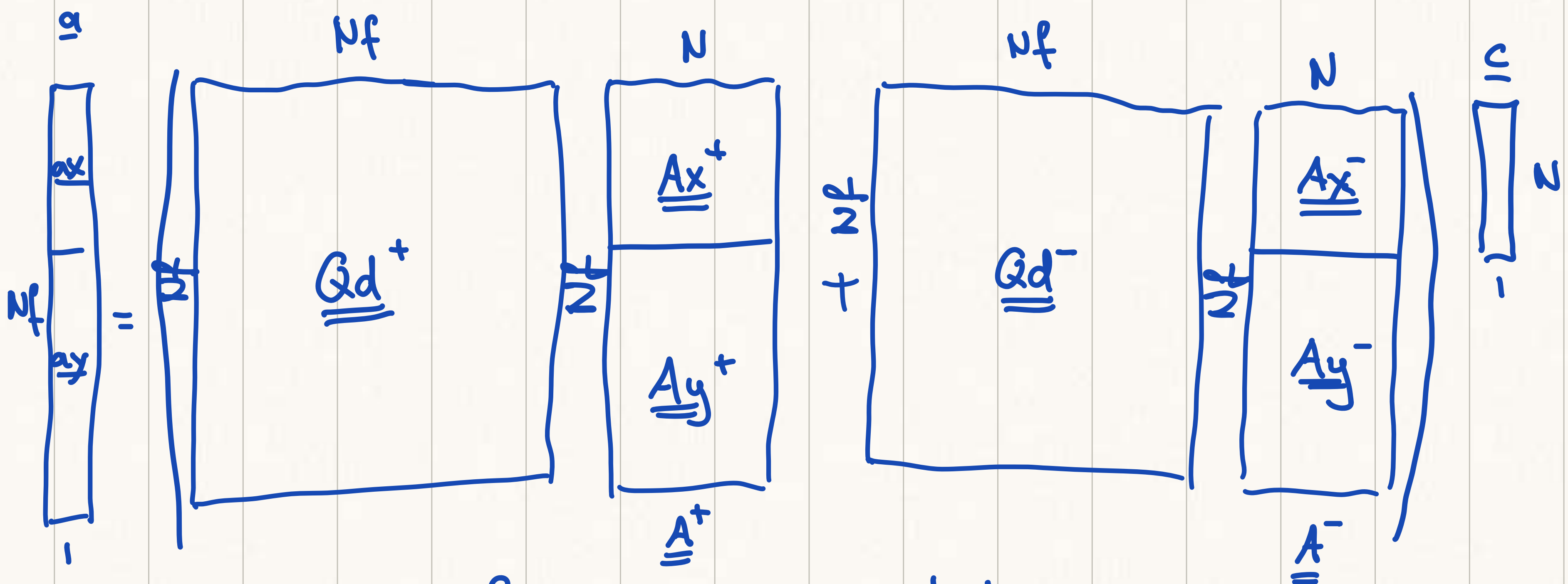
Today: 2D Advection operator

Build 2D Advection Matrix

Problem: In $\underline{\underline{D}}$ and $\underline{\underline{G}}$ the matrix blocks are all identical, but in $\underline{\underline{A}}(\underline{q})$ each block has same structure but different entries, because $\underline{q}$ varies across domain.

Solution: Separate structure, i.e. 0's and 1's, from the magnitudes in $\underline{q}$.

Over all structure of $\underline{\underline{A}}$:



where we have following sparse matrices:

mag. $\begin{cases} \underline{\underline{Qd}}^+ = Nf \times Nf \text{ matrix with pos. fluxes on diagonal} \\ \underline{\underline{Qd}}^- = \text{" " " " neg. " " "} \end{cases}$

structure $\begin{cases} \underline{\underline{A}}^+ = Nf \times N \text{ matrix with ones in locations of pos. fluxes} \\ \underline{\underline{A}}^- = \text{" " " " " " " " neg. "} \end{cases}$

If the flow is evolving only $\underline{\underline{Qd}}^+$ and $\underline{\underline{Qd}}^-$ have to be updated.

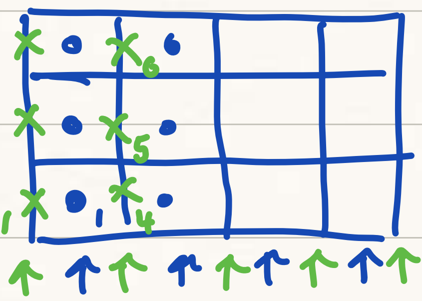
$$\underline{\underline{Qdp}} = \mathrm{spdiags}(\max(q,0),0,Nf,Nf)$$
$$\underline{\underline{Qdn}} = \mathrm{spdiags}(\min(q,0),0,Nf,Nf)$$

So that: $\boxed{\underline{\underline{A}}(q) = \underline{\underline{Qdp}}(q)\,\underline{\underline{Ap}} + \underline{\underline{Qdn}}(q)\,\underline{\underline{An}}}$

$$\underline{\underline{Ap}} = \begin{bmatrix} \underline{\underline{Axp}} \\ \underline{\underline{Ayp}} \end{bmatrix} \qquad \underline{\underline{An}} = \begin{bmatrix} \underline{\underline{Axn}} \\ \underline{\underline{Ayn}} \end{bmatrix}$$

Main task: Assembly of $\underline{\underline{Axp}}$, $\underline{\underline{Axn}}$, $\underline{\underline{Ayp}}$, $\underline{\underline{Ayn}}$

Ax matrices ($\underline{\underline{Axp}}$ & $\underline{\underline{Axn}}$)



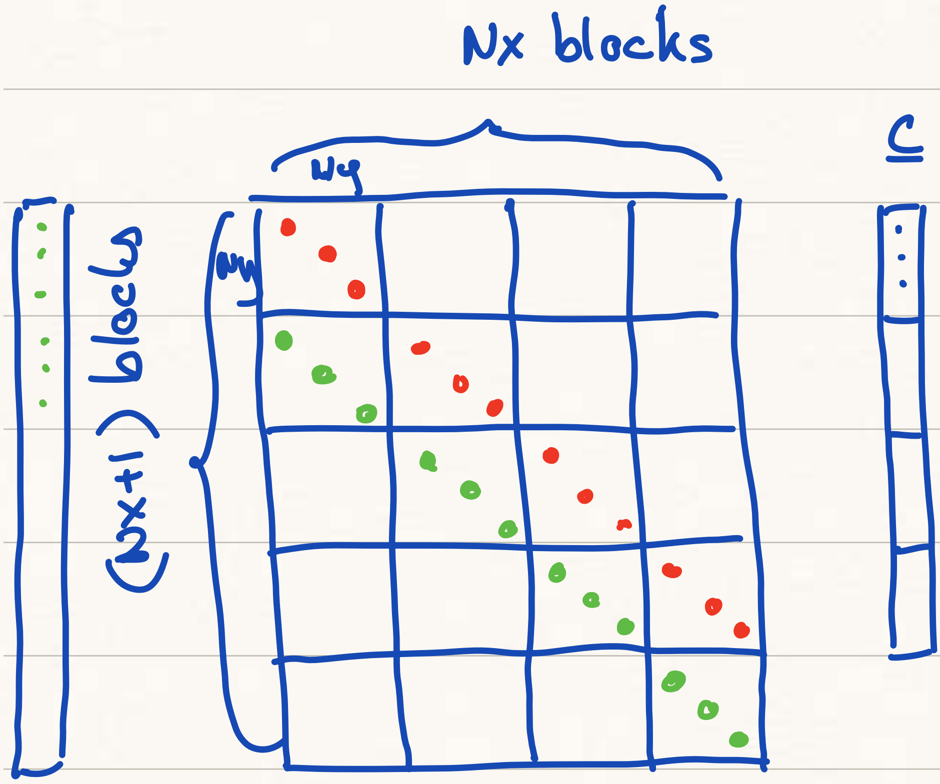
$\underline{\underline{Ax}}$ computes $Ny \times (Nx+1)$ fluxes from Nx by Ny conc.

Nx columns of conc.

$Nx+1$ columns of fluxes

Each block is Ny by Ny

$q<0$ $\quad$ $q>0$



How do we assemble this with kron

$\underline{\underline{C}} \otimes \underline{\underline{D}}$ $\quad\Rightarrow$ $\underline{\underline{C}}$ gives pattern

$\underline{\underline{D}}$ is being copied

The matrix that is copied: $\underline{\underline{I_y}}$ = speye(Ny)

The pattern of blocks is same as in 1D

$\underline{\underline{A}}^+$

$$\begin{bmatrix} & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$\underline{\underline{A}}^-$

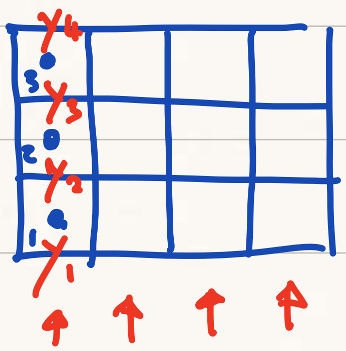
$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}$$

2D $\underline{A_x}$ matrices: $\boxed{\begin{aligned} \underline{\underline{A_{xp}}} &= \mathrm{kron}(\underline{\underline{A_{xp}}}, \underline{\underline{I_y}}) \\ \underline{\underline{A_{xu}}} &= \mathrm{kron}(\underline{\underline{A_{xu}}}, \underline{\underline{I_y}}) \end{aligned}}$

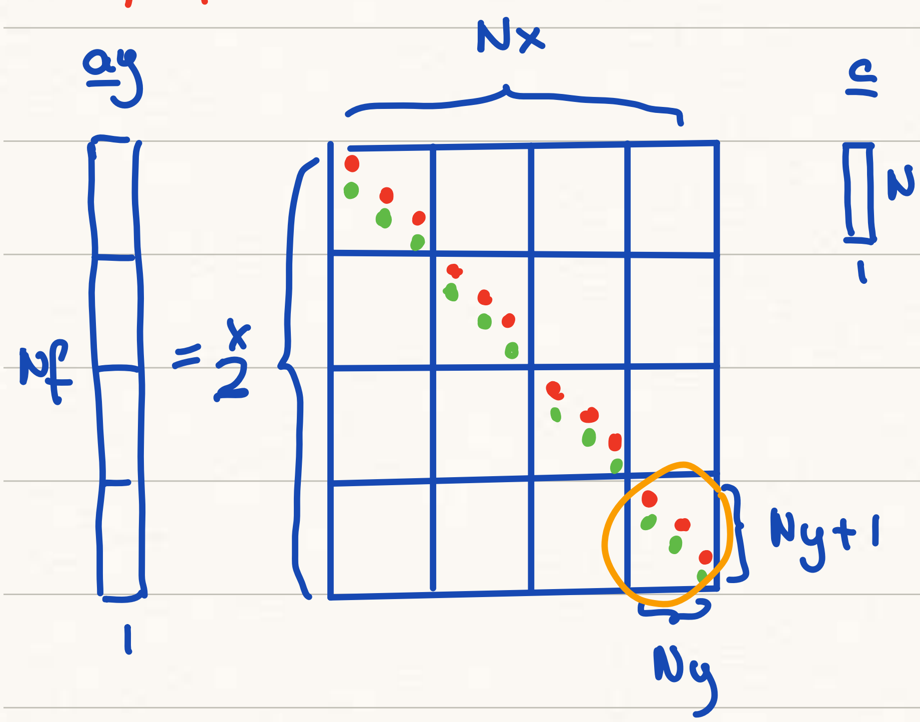
$\uparrow$ 2D $\qquad\qquad \uparrow$ 1D

Ay-matrices

$\underline{\underline{A_y}}$ compute Nx columns of (Ny+1) fluxes from Nx columns of Ny conc.



$\Rightarrow \underline{\underline{A_y}}$ is Nx by Nx block matrix with blocks of size Ny+1 by Ny



block diagonal matrix

Assemble with kron: $\underline{C} \otimes \underline{D}$

$\underline{\underline{I_x}}$ = speye(Nx) $\qquad \underline{\underline{I_x}}$ 1D

Each 1 D block: $A_{yp} = \text{spdiags}(\text{ones}(N_y,1), -1, N_y+1, N_y)$

$A_{yn} = \text{spdiags}(\text{ones}(N_y,1), 0, N_y+1, N_y)$

Assemble 2D matrices:

$$\boxed{\begin{aligned} A_{yp} &= \text{kron}(I_x, A_{yp}) \\ A_{yy} &= \text{kron}(I_x, A_{yn}) \end{aligned}}$$

2D 1D

$$A_p = \begin{bmatrix} A_{xp} \\ A_{yp} \end{bmatrix} \qquad A_n = \begin{bmatrix} A_{xn} \\ A_{yn} \end{bmatrix}$$

$$A(q) = Qdp(q) * A_p + Qdn(q) * A_n$$