Lecture 22: Unsaturated Flow

Logistics:
- HW 7 is due
- HW 8 will be posted due Apr 11
- HW 6 last chance to submit

Last time: 2D Advection Operator

$$\underline{\underline{A}}(q) = \underline{\underline{G_{dp}}} * \underline{\underline{A_p}} + \underline{\underline{Q_{dn}}}\,\underline{\underline{A_n}}$$

$$\underline{\underline{A_p}} = \begin{bmatrix} \underline{\underline{A_{xp}}} \\ \underline{\underline{A_{yp}}} \end{bmatrix} \qquad \underline{\underline{A_n}} = \begin{bmatrix} \underline{\underline{A_{xn}}} \\ \underline{\underline{A_{yn}}} \end{bmatrix}$$

$$\underline{\underline{A_{xp}}} = \mathrm{kron}(\underline{\underline{A_{xp}}}, \underline{\underline{I_y}})$$

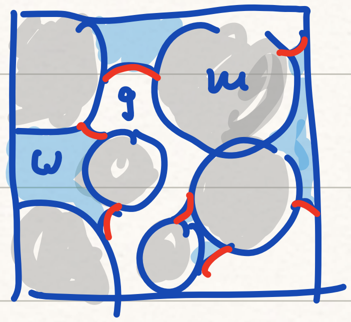
$$\underline{\underline{A_{xn}}} = \mathrm{kron}(\underline{\underline{A_{xn}}}, \underline{\underline{I_y}})$$

$$\underline{\underline{A_{yp}}} = \mathrm{kron}(\underline{\underline{I_x}}, \underline{\underline{A_{yp}}})$$

$$\underline{\underline{A_{yn}}} = \mathrm{kron}(\underline{\underline{I_x}}, \underline{\underline{A_{yn}}})$$

Today: Unsaturated flow

Unsaturated Flow



Three phases:

1) Solid matrix (m)

2) Pore fluid/water (w)

3) Pore gas/air (g)

The water-air interface ⇒ new physics

Volume fractions: $\theta_p = \frac{V_p}{V_T}$ $\quad V_T = V_m + V_f + V_g$

$$0 \le \theta_p \le 1 \qquad \sum_p \theta_p = 1$$

Porosity: $\phi = 1 - \theta_m = \theta_f + \theta_g$

Water content

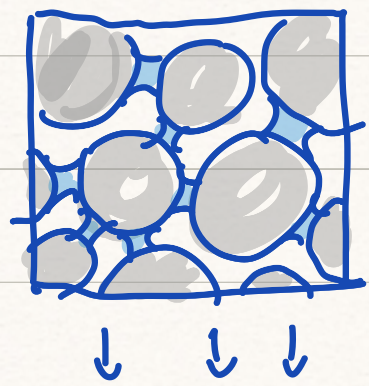
$\theta_w \equiv \theta$ water content

$\theta = \theta(\underline{x}, t)$ ⇒ new dependent variable !

saturated: $\theta = \phi = \theta_s$ $\quad (\theta_g = 0)$

saturated water content

$$\theta \le \theta_s$$



immobile residual water content

θ_r

$\theta_r \leq \theta$

(could evaporate)

$\Rightarrow \quad \theta_r \leq \theta \leq \theta_s$

Rescale water content

$$s = \frac{\theta - \theta_r}{\theta_s - \theta_r}$$

water saturation

(effective water content θ_e, Θ)

Darcy's law for saturated flow

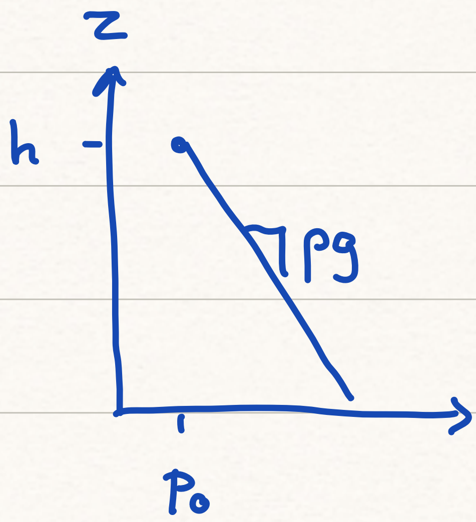
head form $\quad \underline{q} = -K \nabla h$

head is a "potential" for flow, water flows down the potential gradient.

pressure form: $\quad \underline{q} = -\frac{k}{\mu}(\nabla p + \rho g \hat{z})$

p is not potential $\quad \nabla p = 0 \not\Leftrightarrow q = 0$

Relation between h and p



hydrostatic:

$$p = p_0 + \rho g (h - z)$$

ρ = fluid density

$$h = \frac{p - p_0}{\rho g} + z$$

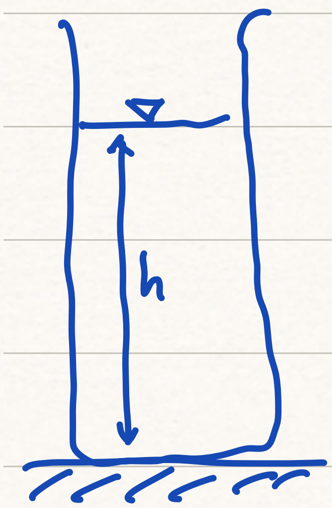
$h = (H = \psi)$ total head or total potential

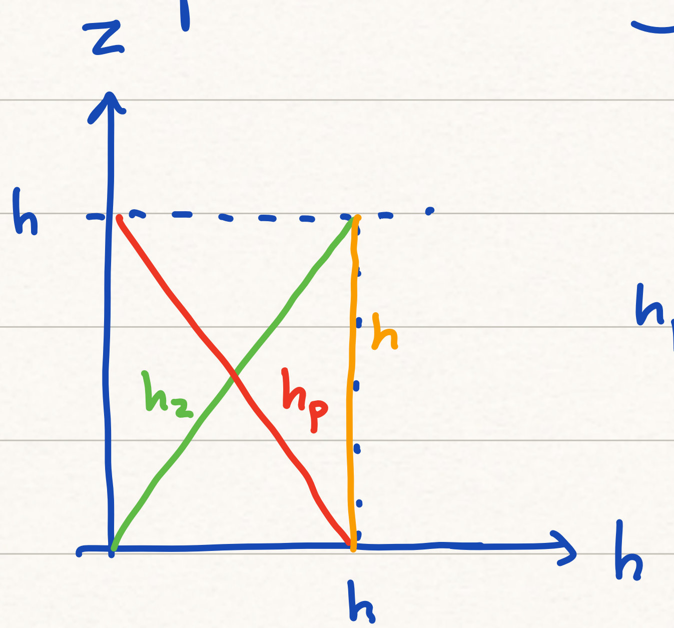
$\frac{p - p_0}{\rho g} = h_p = \psi_p$ pressure head / potential

$z = h_z = \psi_z$ elevation head / grav. potential

only potentials in saturated flow

How do these potentials vary at eqbm





$$h = h_p + h_z$$

$$h_p = \frac{p - p_0}{\rho g}$$

$$= h - z$$

$\nabla h = 0 \Rightarrow q = 0$

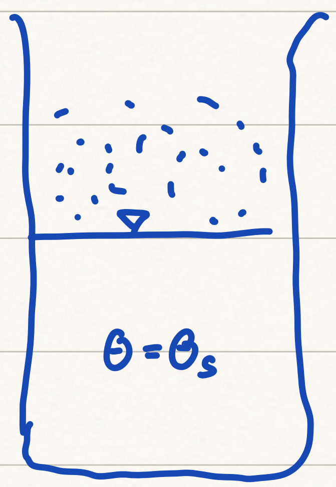
Unsaturated flow

$h = h_m + h_p + h_z$

h_m = matric head/potential → capillary forces

$h_p = 0$ if $\theta < \theta_s$ only in saturated region

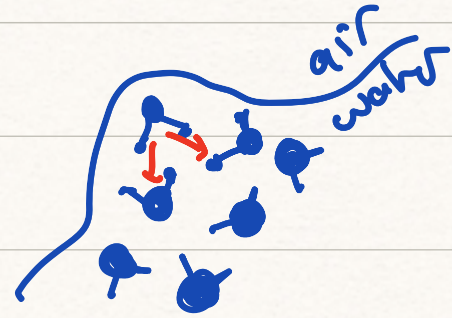
$h_m = 0$ if $\theta = \theta_s$



What is h_m ?

Capillarity & Matric Potential Head

⇒ water-air interface



Surface tension:

near interface hydrogen bonds create a net attraction into the liquid.

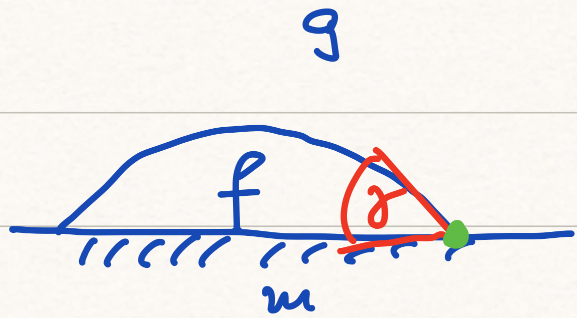
⇒ reduce curvature of the interface

⇒ Energy stored in the interface

surface tension: $\sigma_{wa} = 72.7\ mJ/m^2$

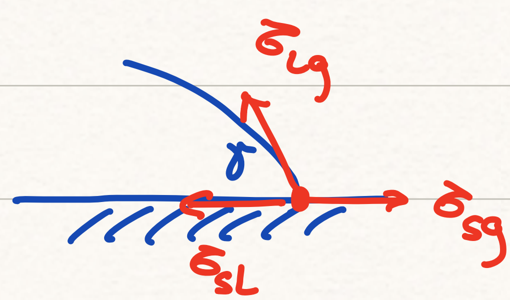
Contact angle:

γ = contact angle at the triple line

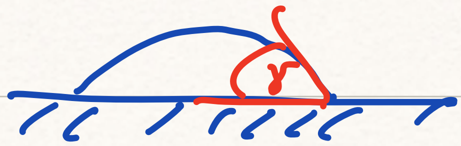


mechanical equilibrium

$$\sigma_{sg} = \sigma_{sL} + \cos\gamma\ \sigma_{Lg}$$



$$\cos\gamma = \frac{\sigma_{sg} - \sigma_{sL}}{\sigma_{Lg}}$$



wetting

$\gamma < 90°$

non-wetting

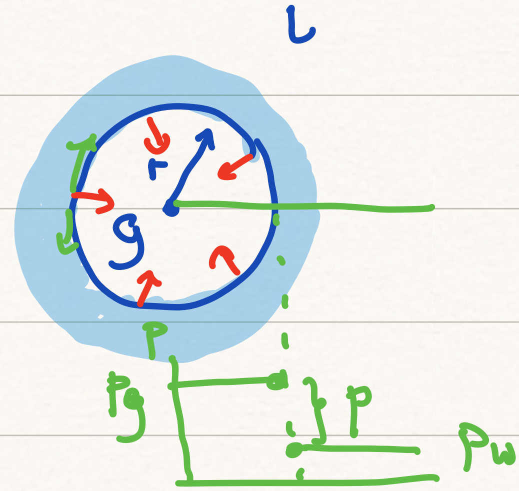
$\gamma > 90°$

⇒ unsaturated flow is strongly dependent on wetting properties

On most mineral surfaces water is wetting fluid relative to air

Capillary pressure

surface tension creates a normal force on concave side.



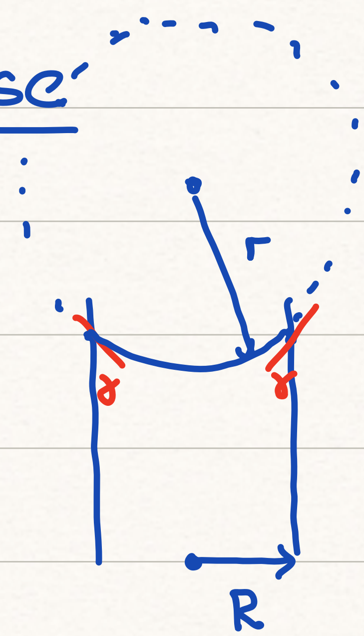
$$p_g - p_w = p_c = \frac{2\sigma}{r}$$

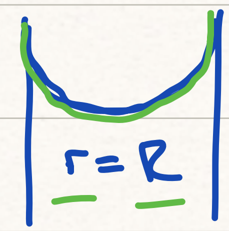
Young-Laplace equ.

$p_c \sim \frac{1}{r}$ r ~ pore radius

$\lim_{r\to\infty} p_c = 0$ → flat interface

Capillary rise





$\gamma = 0$

Given tube radius R and wetting angle γ

$$r = \frac{R}{\cos\gamma}$$

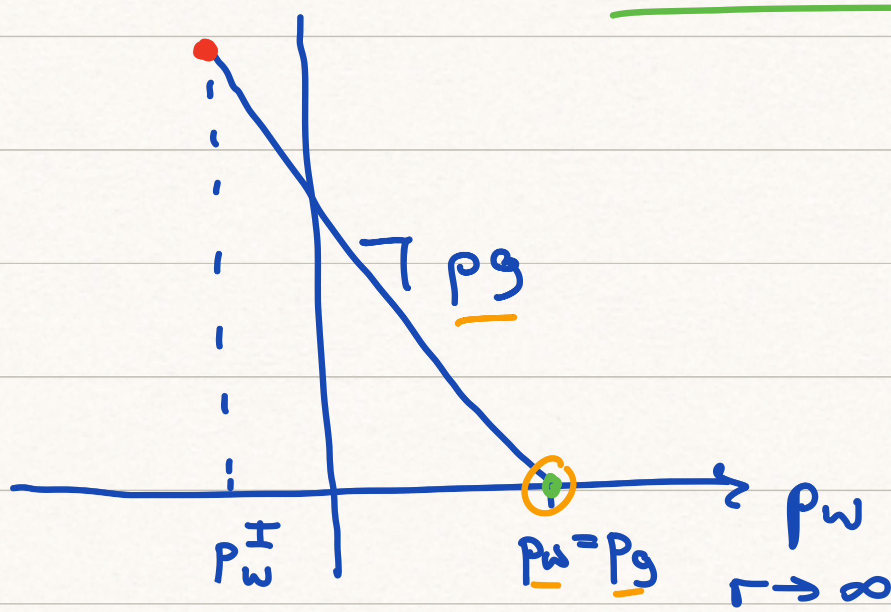
$p_g > p_w$

$p_g \approx p_w$

Curved interface in tube ⇒ pressure jump

Water pressure at interface:

$$p_w^I = p_g - p_c = p_g - \frac{2\sigma}{r} = p_g - \frac{2\sigma\cos\gamma}{R}$$



$$p_w^I + \rho g h = p_g$$

$$p_g - \frac{2\sigma\cos\gamma}{R} + \rho g h = p_g$$

height of capillary rise: $h = \frac{2\sigma\cos\gamma}{\rho g R}$

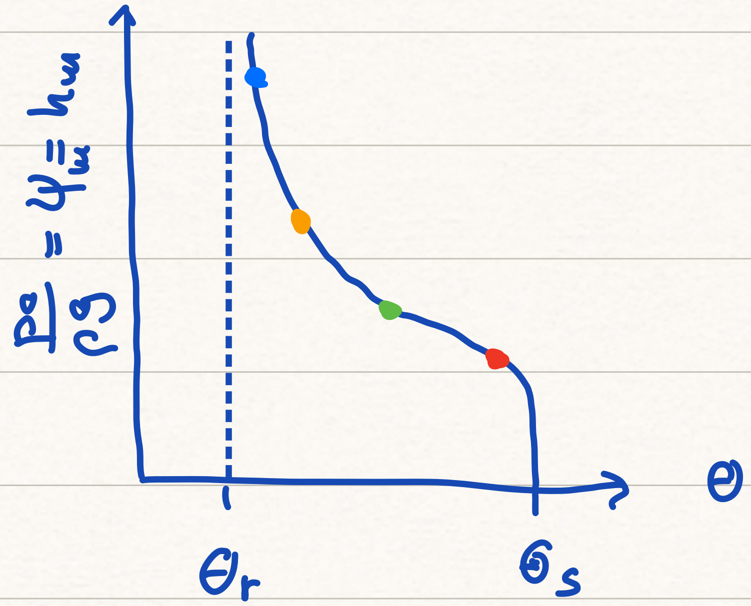
Nomenclature

$$h_m = h = \psi_m = \psi = \frac{p_c}{\rho g}$$

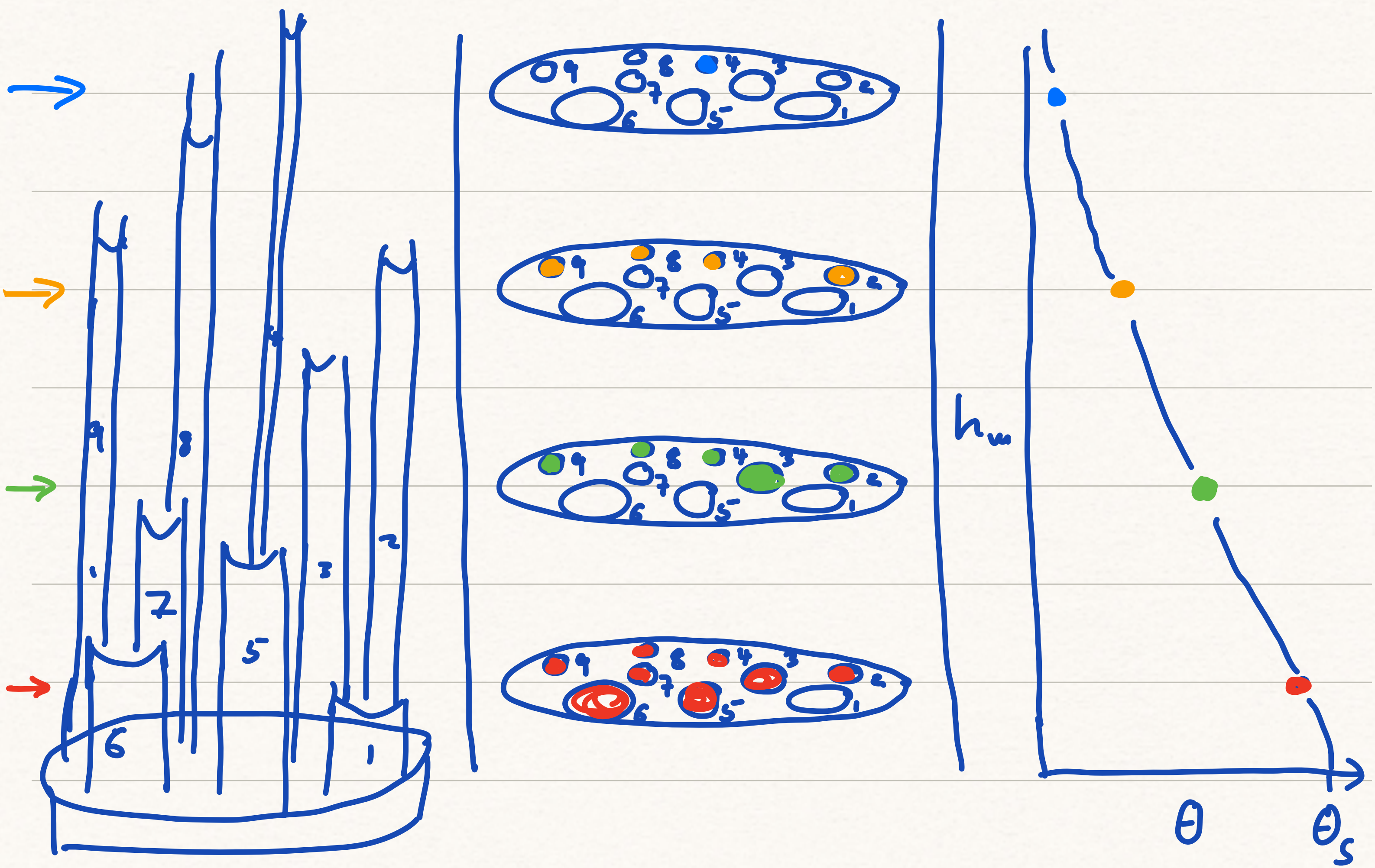
Soil Water Characteristic (SWC)

Relationship between matric head h_m and θ at eqbm.

Primary hydraulic property driving unsaturated flow



Bundle of capillary tubes



Parametric models for SWC

capillary pressure curve

1) Van Genuchten (1980) – VG

$h_m = h = |\psi|$

$$s = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \frac{1}{\left(1 + (\alpha |h|)^n\right)^m}$$

model parameters: α, n, m, θ_r, θ_s

θ_s can be measured independently

often: $m = 1 - \frac{1}{n} = \frac{n-1}{n}$

$$\Rightarrow \quad s = \frac{1}{1 + (\alpha |h|)^{n-1}}$$

3 param: θ_r, α, n

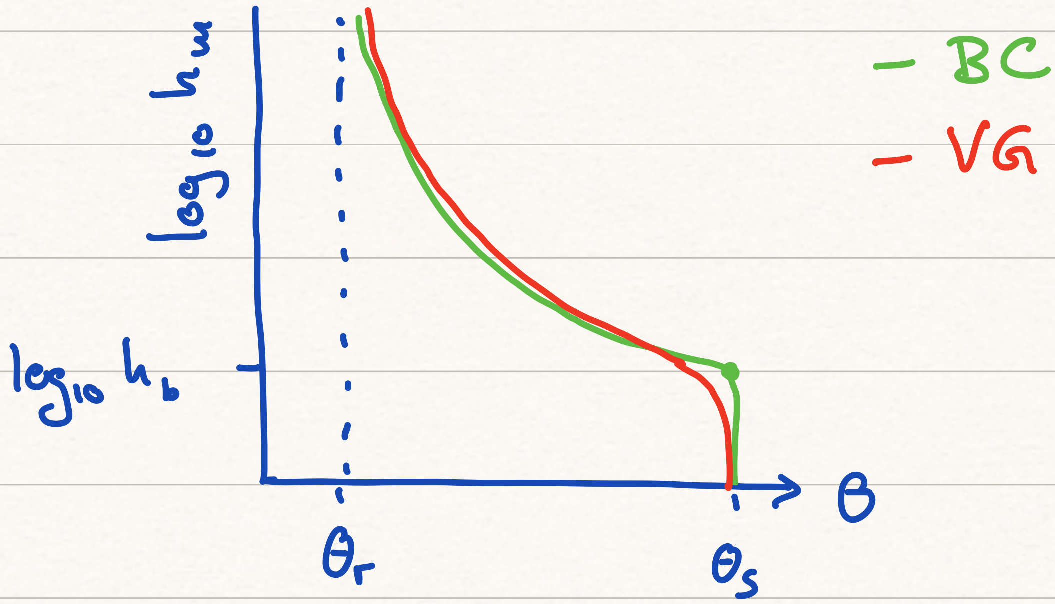
2) Brooks Corey (1964) – BC

$$s = \frac{\theta - \theta_r}{\theta_s - \theta_s} = \left(\frac{h_b}{h}\right)^{\lambda} \qquad h > h_b$$

$$s = 1 \qquad h \leq h_b$$

h_b = is bubbling pressure/ρg

entry pressure



Typical values for a silty loam:

$\theta_s = 0.513$

VG: $\alpha = 0.417 \frac{1}{m}$ $n = 1.75$ $\theta_r = 0.05$

BC: $\lambda = 0.54$ $h_b = 1.48\,m$ $\theta_r = 0.03$

Darcy:

$$q = -K \nabla (h_m + h_z)$$

($h_m \uparrow$ $h(\theta)$; h_z: z)

$$q = -K \nabla (h(\theta) + z)$$

($K \uparrow$ also dependent on θ)

Unsaturated Hydraulic Conductivity

Buckingham (1907)

K_s = saturated hyd. Conductivity

$K_r(s)$ = relative. hyd. "

1) Mualem (1976) - Van Genuchten

$$\frac{K_r(s)}{K_s} = \sqrt{s}\left[1 - \left(1 - s^{\frac{1}{m}}\right)^m\right]^2$$

$$s = \frac{\theta - \theta_r}{\theta_s - \theta_r}$$

$K_r(\theta)$

Can express in terms of matric head

$$\frac{K_r(h)}{K_s} = \frac{\left[\left(1-(\alpha h)\right)^{n-1}\left[1+(\alpha h)^n\right]^{-m}\right]^2}{\left[1+(\alpha h)^n\right]^{\frac{m}{2}}}$$

$$K_r = K_r(\theta) = K_r(h)$$

2) Brooks Corey

$$\frac{K_r(s)}{K_s} = s^{3+\frac{2}{\lambda}} \qquad \frac{K_s(h)}{K_s} = \left(\frac{h}{h_b}\right)^{-2-3\lambda}$$

$\Rightarrow$ Darcy's law:

$$q = -K_r(\theta)\,\nabla(h(\theta) - z)$$
$$= -K_r(h)\,\nabla(h - z)$$

Richards Equation

Mass balance of water

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla\cdot[\rho q] = 0$$

$$\boxed{\frac{\partial \theta}{\partial t} - \nabla\cdot\left[K_r(\nabla h + 1)\right] = 0}$$ mixed form

Use SWC $\nabla h = \frac{dh}{d\theta}\nabla\theta$

$$\frac{\partial\theta}{\partial t} - \nabla\cdot\left[K_r(\theta)\left(\frac{dh}{d\theta}\nabla\theta + \hat{z}\right)\right]$$

introduce $D(\theta) = K_r(\theta)\frac{dh}{d\theta}$ soil water diffusivity

$$\frac{\partial \theta}{\partial t} - \nabla\cdot\left[D(\theta)\nabla\theta + K_r(\theta)\hat{z}\right]$$ saturation form

use SWC $\frac{\partial \theta}{\partial t} = \frac{d\theta}{dh}\frac{\partial h}{\partial t} = c(h)\frac{\partial h}{\partial t}$

↑ soil capacity

$$c(h)\frac{\partial h}{\partial t} - \nabla\cdot\left[K_r(h)(\nabla h + \hat{z})\right]$$ head form

Fluxes in Richards equ

$$\frac{\partial \theta}{\partial t} - \nabla\cdot\left[\underbrace{D(\theta)\nabla\theta}_{\text{diffusive flux}} + \underbrace{K_r(\theta)\hat{z}}_{\text{advective flux}}\right] = 0$$

diffusive flux → capillarity

advective flux → gravity

non-linear advection diffusion equ