# Lecture 23: Solving Non-Linear Equations

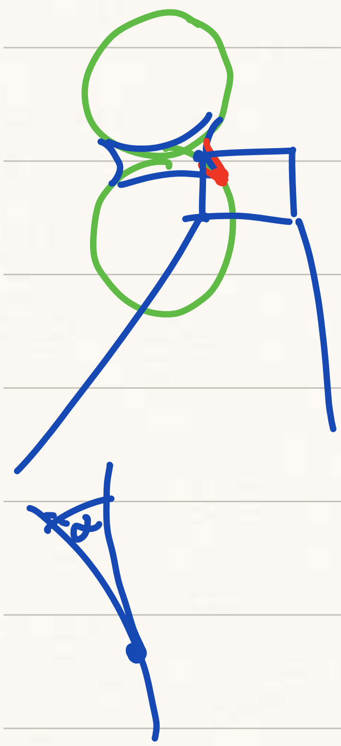
Logistics: - HW8 due Thursday

- HW6 last chance

Last time: - Intro to unsaturated flow

- Water content $\theta(\underline{x},t)$
- water saturation $s = \frac{\theta-\theta_r}{\theta_s-\theta_r}$ $\quad 0 \le s \le 1$

  $\uparrow \phi$



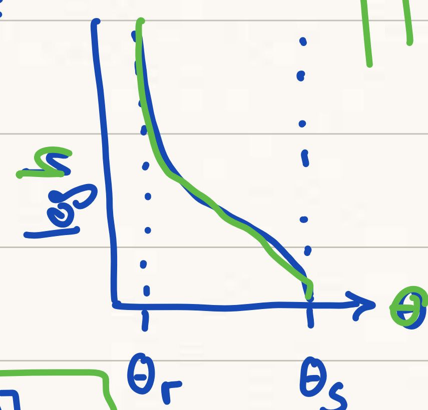
- Capillarity
  - surface tension $\sigma$
  - contact angle $\cos\gamma = \frac{\sigma_{sg}-\sigma_{sL}}{\sigma_{lg}}$
  - capillary pressure: $p_g - p_w = \frac{2\sigma}{r}$ Y-L

⇒ Capillary rise : $h = \frac{2\sigma\cos\gamma}{\rho g R}$

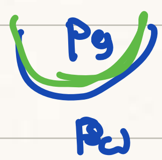
matric potential head

- Soil water Characteristic

  $h = h(\theta)$




- Unsaturated $K = K(\theta)$

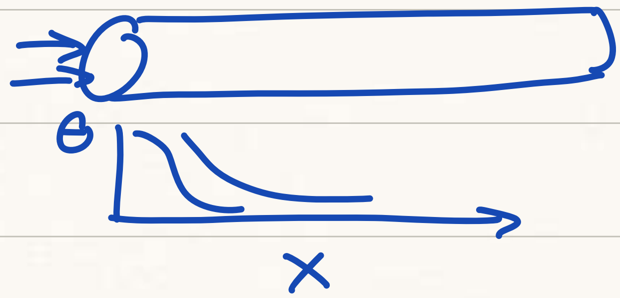



⇒ Richards Eqn

$$\boxed{\frac{\partial\theta}{\partial t} - \nabla\cdot\left[K(\theta)\left(\nabla h + \underline{z}\right)\right] = 0}$$

Today: How to solve?

# Richards equ for horizontal domain



$$\Rightarrow \quad \frac{\partial \theta}{\partial t} - \nabla \cdot \left[ K(\theta) \nabla h \right] = 0$$

$h = h(\theta)$

$$\frac{\partial \theta}{\partial t} - \nabla \cdot \Big[ \underbrace{K(\theta) \frac{dh}{d\theta}}_{D(\theta)} \nabla \theta \Big]$$

$$\boxed{\frac{\partial \theta}{\partial t} - \nabla \cdot \underbrace{\left[ D(\theta) \nabla \theta \right]}_{\text{non-lin.}} = 0}$$

Diffusion equation: $\quad \frac{\partial c}{\partial t} - \nabla \cdot \left[ D \nabla c \right] = 0$

$c \rightarrow \theta$

$\Rightarrow$ non-linear diffusion equation.

# Solving non-linear equations

$$\underline{\underline{I}}\,\frac{\underline{\theta}^{n+1}-\underline{\theta}^{n}}{\Delta t} - \underline{\underline{D}} * [\,\underline{\underline{Kd}}(\underline{\theta}) * \underline{\underline{G}}\,\underline{\theta}\,] = \underline{0}$$

$$\underline{\underline{I}}(\underline{\theta}^{n+1}-\underline{\theta}^{n}) \underbrace{- \Delta t\, \underline{\underline{D}}[\,\underline{\underline{Kd}}(\underline{\theta}^{n}) * \underline{\underline{G}}\,]}_{\underline{\underline{L}}\,\text{ij}}\, \underline{\theta}^{n} = \underline{0}$$

## Explicit time integration

$\underline{\theta} = \underline{\theta}^{n}$

$$\Rightarrow \quad \underbrace{\underline{\underline{I}}}_{\underline{\underline{IM}}}\,\underline{\theta}^{n+1} = \underbrace{[\underline{\underline{I}} - \Delta t\, \underline{\underline{L}}(\underline{\theta}^{n})]}_{\underline{\underline{EX}}(\underline{\theta}^{n})}\,\underline{\theta}^{n}$$

$$\underline{\underline{L}}(\theta) = -\underline{\underline{D}} * \underline{\underline{Kd}}(\theta) * \underline{\underline{G}}$$

$\underline{\underline{Kd}}$ ↳ $D(\theta)$ hyd. diffusivity.

$$\underline{\underline{IM}}\,\underline{\theta}^{n+1} = \underline{\underline{EX}}(\underline{\theta}^{n})\,\underline{\theta}^{n} + \Delta t\, \underline{f}_s$$

$\Rightarrow$ solve with solve_lbvp.m

$$\Delta t \le \frac{\Delta x^2}{2\max(D(\theta))}$$ timestep restriction!

## Implicit time integration

↳ lag the non-linearity

$$\underline{\underline{I}}\,(\underline{\theta}^{n+1}-\underline{\theta}^{n}) + \Delta t\,\underline{\underline{L}}(\underset{\uparrow}{\underline{\theta}^{n}})\,\underline{\theta}^{n+1} = \underline{0}$$

$$\underbrace{\underline{\underline{I}} + \Delta t\,\underline{\underline{L}}(\underline{\theta}^{n})}_{\underline{\underline{IM}}(\underline{\theta}^{n})}\,\underline{\theta}^{n+1} = \underbrace{\underline{\underline{I}}}_{\underline{\underline{EX}}}\,\underline{\theta}^{n}$$

$$\underline{\underline{IM}}(\underline{\theta}^{n})\,\underline{\theta}^{n+1} = \underline{\underline{EX}}\,\underline{\theta}^{n}$$

## Solve fully non-linear

$$\underline{\underline{I}} + \Delta t\,\underline{\underline{L}}(\underline{\theta}^{n+1})\,\underline{\theta}^{n+1} = \underline{\underline{I}}\,\underline{\theta}^{n}$$

$$\underline{N}(\underline{\theta}^{n+1}) = \underline{\theta}^{n}$$

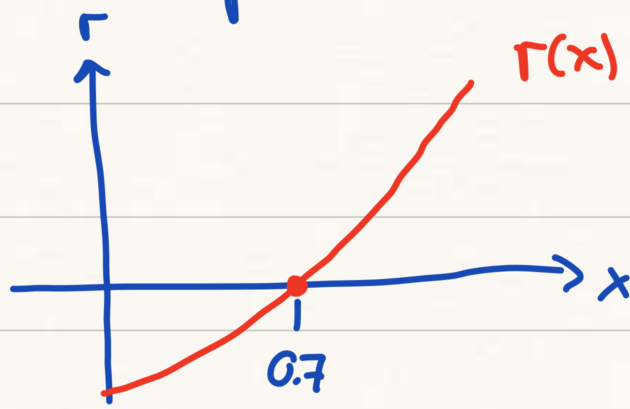
↑ non-linear vector-valued vector function

⇒ system of non-linear algebraic eqs

⇒ Newton-Raphson Method !

# Newton-Raphson Method

Suppose we have simple non-lin. function




$$r(x) = e^x - 2$$

and we want to find

$r(x) = 0$ root

$x = \ln(2) \approx 0.7$ (analytic)

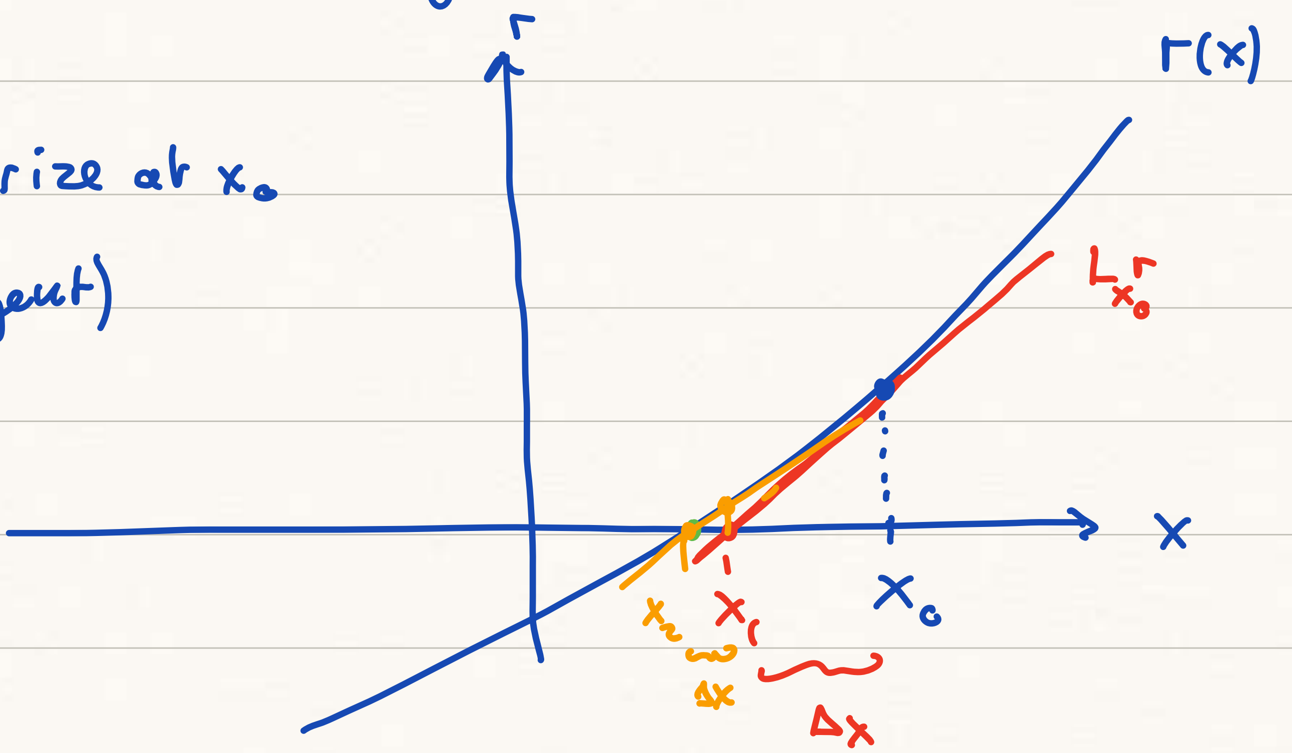
Numerical solution requires iteration

Sequence of improving approximations

$\Rightarrow$ need an inital guess, $x_0$ !

linearize at $x_0$

(tangent)



linearized equation: → Taylor series

$$L_{x_0} r = r(x_0) + \frac{dr}{dx}\bigg|_{x_0} \underbrace{(x - x_0)}_{\Delta x} + O(\Delta x^2)$$

Root of $L_{x_0} r$: $\quad r(x_0) + \frac{dr}{dx}\Big|_{x_0} \Delta x = 0$

$$\Rightarrow \Delta x = -r(x_0) \Big/ \underbrace{\frac{dr}{dx}\Big|_{x_0}}_{J(x_0)}$$

root: $x_1 = x_0 + \Delta x$

Newton-Raphson turns this into an iterative method that converges to root of $r(x)$ quadratically.

k-th iteration

$$\boxed{\begin{aligned} \Delta x^k &= -r(x^k) \Big/ \frac{dr}{dx}\Big|_{x^k} \\ x^{k+1} &= x^k + \Delta x^k \end{aligned}}$$