Lecture 3: Intro to numerics   5.220E

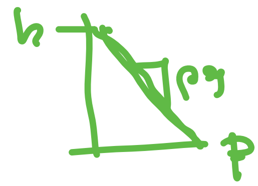
Logistics: - Mbarak office hours (11-noon Wed)

- test if you can see HW 1 problems

- Last time: Balance laws

accumulation   fluxes   source term

- General balance law: $\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = \hat{f}_s}$
- pore fluid mass balance: $\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho \underline{q}) = \hat{f}_s$
- constitutive laws:



1) Darcy's law : $\underline{q} = -K\,\nabla h$

2) Equation of state: $\rho = \rho(h)$ h = head

$h \sim p$

- Divergence & Gradient:
  - gradient: ∇ scalar → vector
  - divergence: $\nabla\cdot$ vector → scalar

Today: - Finish incompressible flow

⇒ Motivate our approach to numerics

- Finite differences
- Differentiation matrices

Example: Flow around well

→ Conservative Finite Differences

Incompressible Flow

Fluid mass balance: $\frac{\partial}{\partial t}(\phi\rho) + \nabla\cdot(\rho\,\underline{q}) = \hat{f}_s = \rho f_s$

- porosity does not change with time $\phi \neq \phi(t)$

  but $\phi = \phi(\underline{x})$

- for pressure variations in infiltration problem

  ρ = const.

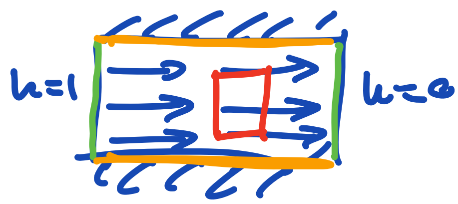
Substitute them:

$$\cancel{\frac{\partial}{\partial t}(\phi\rho)}^{\,0} + \cancel{\rho}\,\nabla\cdot \underline{q} = \cancel{\rho}\, f_s$$

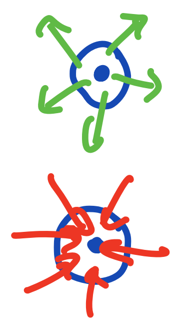
$$\nabla\cdot \underline{q} = f_s$$

$$\nabla\cdot \underline{q} = 0$$

$f_s = 0$



$f_s > 0$



$f_s < 0$

substitute Darcy's law:

$$\left.\begin{aligned} \nabla\cdot \underline{q} &= f_s \\ \underline{q} &= -K\nabla h \end{aligned}\right\} \quad \boxed{-\nabla\cdot(K\nabla h) = f_s}$$

Poisson's Equ

if K = const.

$$-K\,\nabla\cdot\nabla h = f_s$$

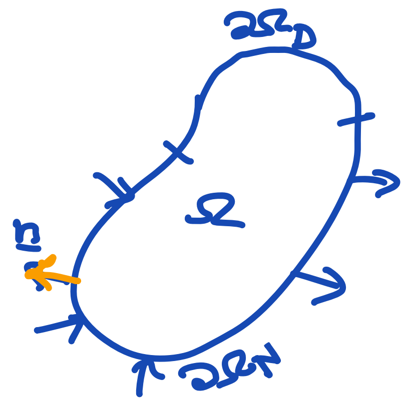
$$-K\,\nabla^2 h = f_s \qquad \nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}$$

Boundary value problem (BVP)

A well-posed problem requires boundary conditions (BC)

PDE: $\nabla \cdot \underset{\sim}{q} = f_s$

$\underset{\sim}{q} = -K \nabla h$



BC: a) Dirichlet BC

prescribe the solution

$$h(\underset{\sim}{x}) = h_B(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \partial\Omega_B$$

b) Neuman BC

prescribe the flux

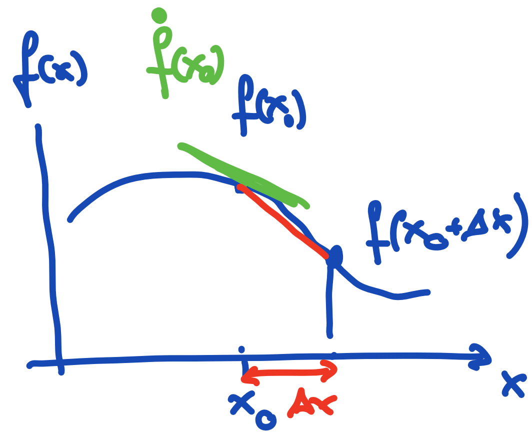
$$\underset{\sim}{q} \cdot \underset{\sim}{n} = -q_B \qquad \underset{\sim}{x} \in \partial\Omega_N$$

$q_B >$ inflow

$q_B <$ outflow

$q_B = 0$ no flow (natural BCs)

Intro to Finite Differences



In calculus we define the derivative of function $f(x)$

as $$\dot f(x_0) = \left.\frac{df}{dx}\right|_{x_0} = \underbrace{\lim_{\Delta x \to 0}}_{\text{don't take limit}} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

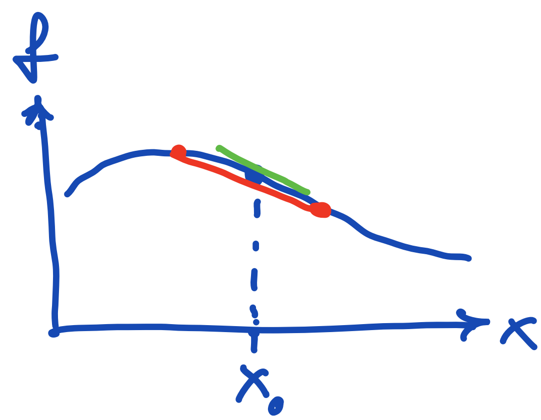
Finite differences (one-sided difference)

$$\dot f(x_0) \approx \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$\dot f(x_0) = \quad '' \quad + O(\Delta x)$$

In proper numerics class you prove that error Δx

$\Rightarrow$ first-order accurate



Central difference

$$\dot f(x) = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}$$

$\Rightarrow$ second order accurate

Differentiation Matrix

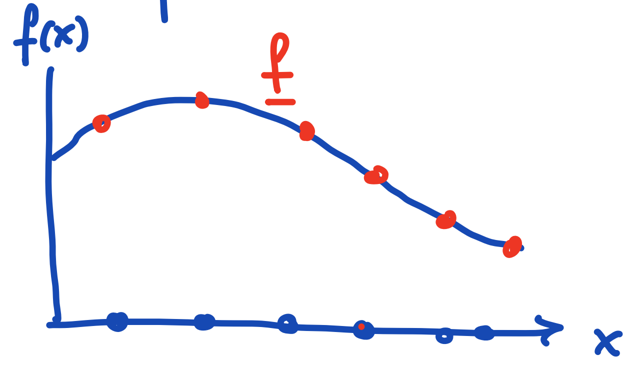
Derivative is a linear differential operator

it takes a function and returns different function

$$\dot{f}(x) = \mathcal{D}(f(x))$$

↑ derivative operator

The discrete equivalent of $f(x)$ is a vector.
$\underline{f} = f(\underline{x})$. Similarly
we can define $\underline{df} = \dot{f}(\underline{x})$



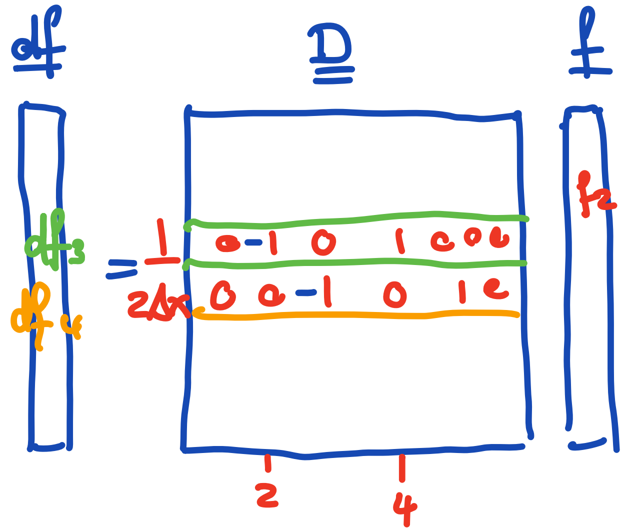
What is the discrete equivalent of $\mathcal{D}$?

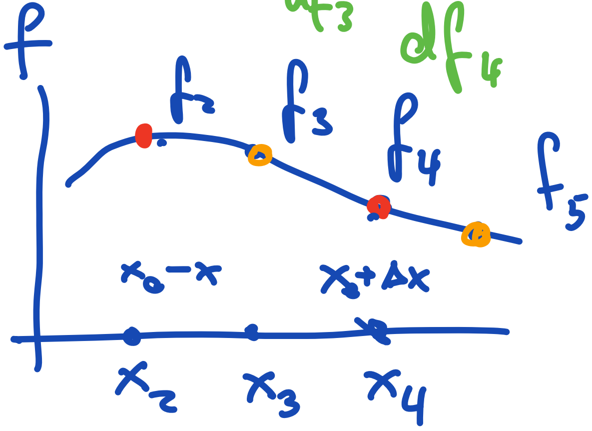
$$\underline{df} = \boxed{\underline{D}}\,\underline{f}$$

has to be a matrix, because it is linear
and it relates two vectors.

⇒ Differentiation matrix

df





$$df_3 = \frac{f_4 - f_2}{2\Delta x}$$

$$\begin{bmatrix} -1 & & 1 & & & & \\ & -1 & & 1 & & & \\ & & -1 & & 1 & & \\ & & & -1 & & 1 & \\ & & & & -1 & & \end{bmatrix}$$

$\Rightarrow$ $\underline{\underline{D}}$ has simple bi-diagonal structure

Note: Boundaries require different treatment.

What about 2nd derivatives? $\frac{d^2 f}{dx^2}$

$$\underline{ddf} = \underline{\underline{D}}\,\underline{df} = \underline{\underline{D}}\,\underline{\underline{D}}\,\underline{f} = \underline{\underline{D}}^2\,\underline{f}$$