Lecture 4: Conservative Finite Differences

Logistics: Office hours: Marc — Tu 3-4 pm JGB 4.216G

Mbarak — Wed 11am - noon JGB 5.220E

Matlab Grader → HW 1 is posted

Last time: - Incompressible flow $\phi, \rho = \text{const}$

$$\Rightarrow \left.\begin{array}{ll} 1) & \nabla \cdot \underline{q} = f_s \\ 2) & \underline{q} = -K \nabla h \end{array}\right\} \quad -\nabla \cdot (K \nabla h) = f_s$$

- BC: Dirichlet : prescribe h

  Neumann : prescribe flux q

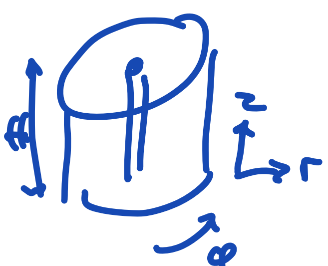
- Finite differences: - one side 1st order (Δx)

  - central diff 2nd order (Δx^2)

- Differentiation matrix: $\underline{df} = \underline{\underline{D}}\ \underline{f}$

  $\underline{ddf} = \underline{\underline{D}}^2\ \underline{f}$

Example: Flow around injection well (cylindrical coord)



cylindrical coord: $\nabla \cdot \underline{q} = \frac{1}{r}\frac{d}{dr}(r\, q_r)$ $\quad \underline{q} = \begin{pmatrix} q_r \\ q_z \\ q_\theta \end{pmatrix}$

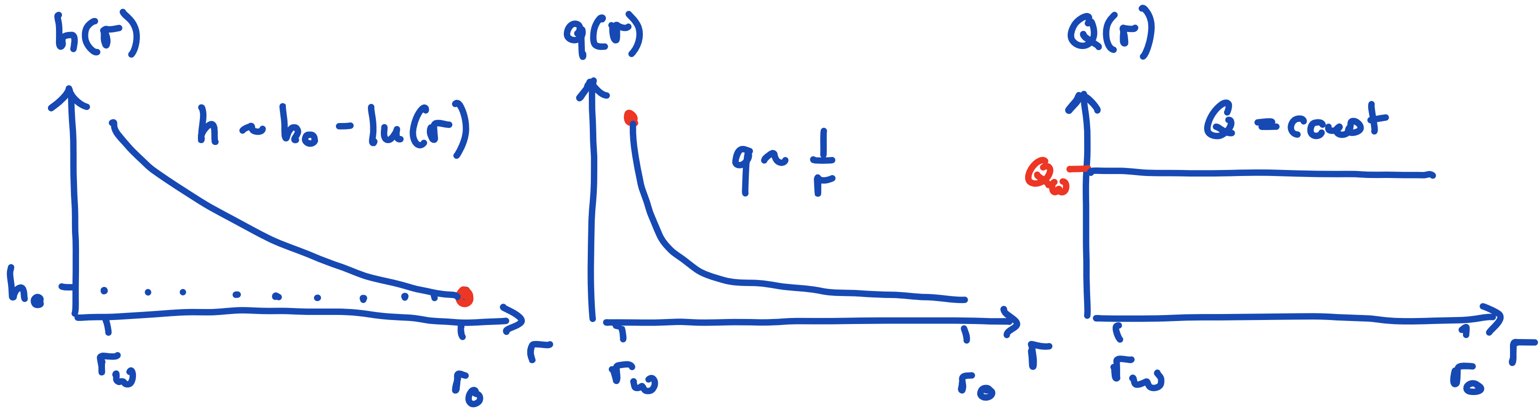
PDE: $-\cancel{\frac{1}{r}}\frac{d}{dr}\left(r K \frac{dh}{dr}\right) = 0 \qquad r \in [r_w \quad r_0]$

BC: $Q_w = A_w\, q_r(r_w) = -A_w K \left.\frac{dh}{dr}\right|_{r_w} \Rightarrow \left.\frac{dh}{dr}\right|_{r_w} = \frac{-Q_w}{A_w K}$

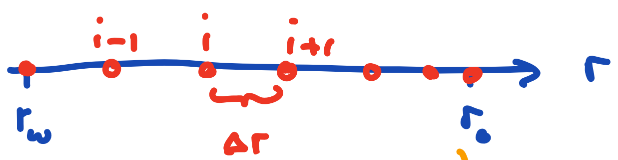
$h(r_0) = h_0$ Dir

$A_w = 2\pi r_w H$

Solution



Finite Difference Approximation



$\nabla\cdot(k\nabla h)$

PDE: $\frac{d}{dr}\left(r\,\cancel{k}\,\frac{dh}{dr}\right) = 0$

$\underline{\underline{D}} \qquad \underline{\underline{D}}$

$r\frac{d^2h}{dr^2} + \frac{dh}{dr} = 0$

$\underline{\underline{R}} = \begin{pmatrix} r_1 & & \\ & r_2 & \\ & & r_3 \end{pmatrix}$

$\underline{\underline{R}}\,\underline{\underline{D}}^2\underline{h} + \underline{\underline{D}}\,\underline{h} = \underbrace{(\underline{\underline{R}}\,\underline{\underline{D}}^2 + \underline{\underline{D}})}\,\underline{h} = \underline{0}$

$\underline{\underline{L}}\,\underline{h} = \underline{0}$ + BC

Key to discrete mass conservation is to discretize conservation form with divergence intact

$$-\nabla\cdot(k\nabla h) = f_s$$

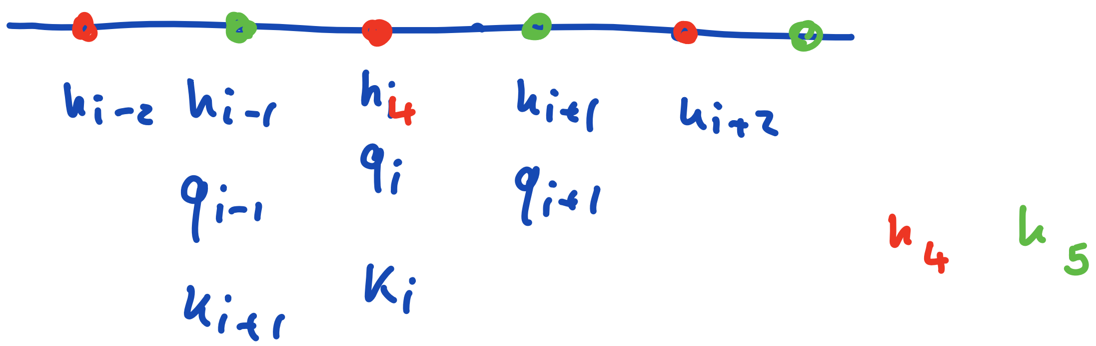
$$-\frac{d}{dx}\left(k\,\frac{dh}{dx}\right) = f_s$$

Break it up into 'div-grad' system two first order equations by introducing flux q:

1) $\nabla \cdot q = f \xrightarrow{1D} \dfrac{dq}{dx} = f \rightarrow \dfrac{q_{i+1} - q_{i-1}}{2\Delta x} = f_i$

2) $q = -k \nabla h \rightarrow q = -k \dfrac{dh}{dx} \rightarrow q_i = -K_i \dfrac{h_{i+1} - h_{i-1}}{2\Delta x}$

if we co-locate ~~grid~~ h and q on grid



substitute q_{i+1} and q_{i-1} in mass balance

$$\frac{1}{2\Delta x}\Bigg(\underbrace{-k_{i+1}\frac{h_{i+2} - h_i}{2\Delta x}}_{q_{i+1}} + \underbrace{k_{i-1}\frac{h_i - h_{i-2}}{2\Delta x}}_{-q_{i-1}}\Bigg) = f_i$$

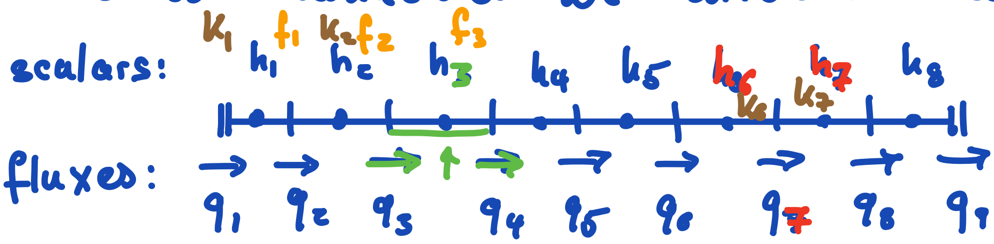
$$\frac{1}{4\Delta x^2}\left[-k_{i+1}h_{i+2} - (k_{i-1} + k_{i+1})h_i - k_{i-1}h_{i-2}\right] = f_i$$

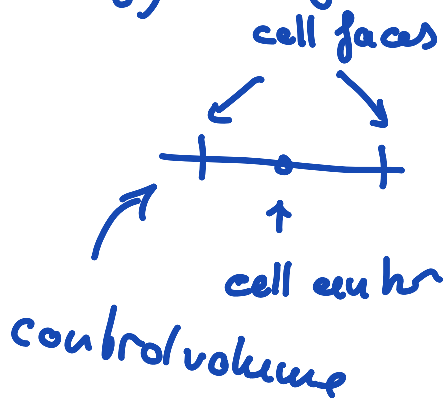
$\Rightarrow$ wide stencil we don't use direct neighbors

$\Rightarrow$ oscillatory solution because even & odd not properly coupled.

Conservative Finite Differences / Finite Volumes

To reduce the width of FD stencil and to couple even and odd unknowns we introduce a staggered grid.





Discretize div-grad system:

1) $\nabla \cdot q = f_s \xrightarrow{1D} \dfrac{dq}{dx} = f \xrightarrow{CFD} \dfrac{\overset{q_4}{q_{i+1}} - \overset{q_3}{q_i}}{\Delta x} = f_i$

2) $q = -k \nabla h \xrightarrow{1D} q = -k \dfrac{dh}{dx} \xrightarrow{CFD} q_i = -\underset{\substack{\uparrow \\ \text{suitable average}}}{k_{i-\frac{1}{2}}} \dfrac{h_i - h_{i-1}}{\Delta x}$

Substitute (2) into (1)

$$-\frac{1}{\Delta x}\left[k_{i+\frac{1}{2}} \frac{h_{i+1} - h_i}{\Delta x} - k_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}\right] = f_i$$

simplify

$$-\frac{1}{\Delta x^2}\left[k_{i+\frac{1}{2}} h_{i+1} - (k_{i+\frac{1}{2}} + k_{i-\frac{1}{2}}) h_i + k_{i-\frac{1}{2}} h_{i-1}\right] = f_i$$

$\Rightarrow$ narrow/compact stencil