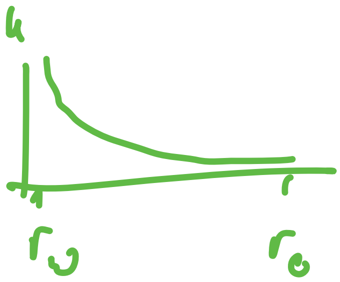
Lecture 5: Discrete Operators

Logistics: - HW 1 is due Feb 8 (next Thursday)

- Office hrs today 3-4 pm JGB 4.216 G

Last time: - Example of flow around well



$\Rightarrow$ Problems due to boundary layer near the well

- Standard FD $\Rightarrow$ large errors & lack of mass conservation

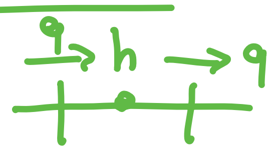
$q \sim \frac{dh}{dx}$

$\nabla \cdot q = f$

- Discretized conservative form

$\Rightarrow$ wide stencil $\Rightarrow$ oscillations

- Staggered grid



$\Rightarrow$ compact stencil no oscillations

Today: - Discrete Operators

- Gradient
- Divergence

- Example of shallow aquifer

Discrete operators:

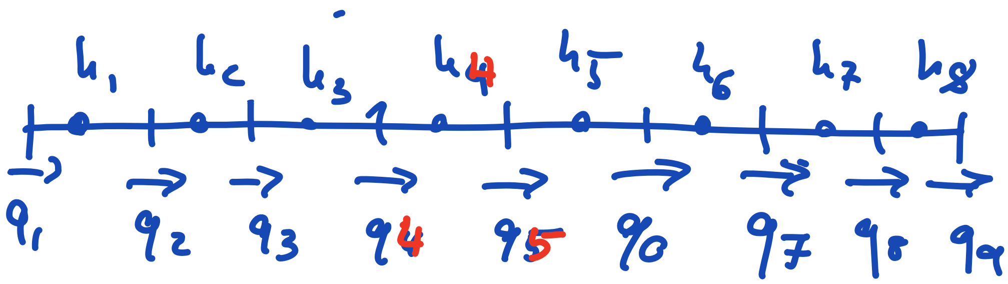
continuous ($K=1$)   discrete

$$\nabla \cdot q = f_s \quad \Rightarrow \quad \underline{\underline{D}} * \underline{q} = \underline{f_s}$$

$$q = -\nabla h \quad \Rightarrow \quad q = -\underline{\underline{G}} * \underline{h}$$

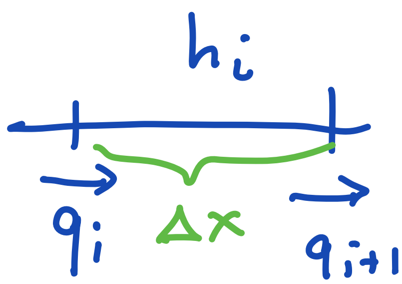
$$\nabla \cdot \nabla h = f_s \quad \Rightarrow \quad -\underline{\underline{D}}\,\underline{\underline{G}}\,\underline{h} = \underline{f_s}$$

$\Rightarrow$ Staggered grid



$N_x = 8$

$N_{fx} = N_x + 1 = 9$



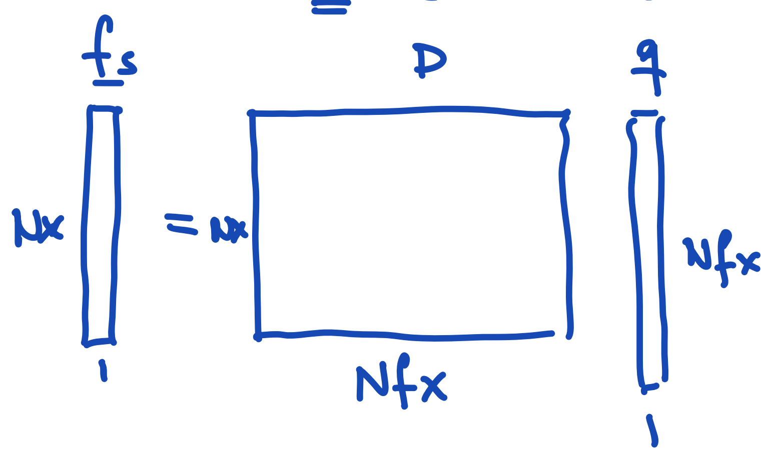
Divide domain 0 to L into N_x cells

$$\Delta x = \frac{L}{N_x}$$

Discrete Divergence Operator

Divergence takes flux and returns scalar:

$$\nabla \cdot \underset{N_{fx}}{q} = \underset{N_x}{f_s} \quad \Rightarrow \quad \underline{\underline{D}} \text{ cannot be square}$$

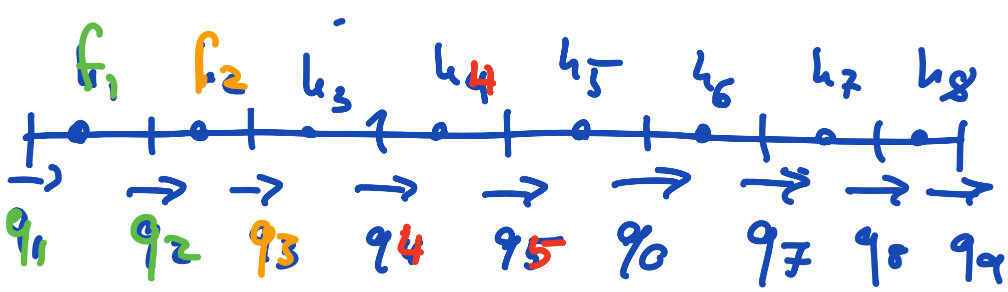


$\Rightarrow$ size of $\underline{\underline{D}}$ is N_x by N_{fx} $(N_x + 1)$

Entries of $\underline{\underline{D}}$

$$\underset{f_s}{\begin{bmatrix} f_1 \\ \vdots \end{bmatrix}} = \frac{1}{\Delta x} \underset{\underline{\underline{D}}}{\begin{bmatrix} -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ & & -1 & 1 & & & & & \\ & & & -1 & 1 & & & & \\ & & & & -1 & 1 & & & \\ & & & & & -1 & 1 & & \\ & & & & & & -1 & 1 & \\ & & & & & & & -1 & 1 \end{bmatrix}} \underset{q}{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ \vdots \end{bmatrix}}$$

$$f_1 = \frac{q_2 - q_1}{\Delta x}$$



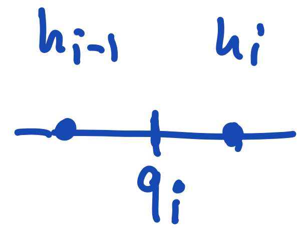
Discrete gradient operator

Gradient takes a scalar (head) and returns flux

$$\underline{q} = -\nabla h \rightarrow \underline{q} = -\underline{\underline{G}}\,\underline{h}$$

$$\begin{array}{ccccc} & \underline{q} & & \underline{\underline{G}} & \underline{h} \\ N_{fx} & \begin{bmatrix} \\ \\ \\ \end{bmatrix} & = \; N_{fx} & \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{N_x} & \begin{bmatrix} \\ \\ \\ \end{bmatrix} N_x \end{array}$$

$\Rightarrow \underline{\underline{G}}$ is N_{fx} by N_x matrix



$$q = -\frac{h_i - h_{i-1}}{\Delta x}$$

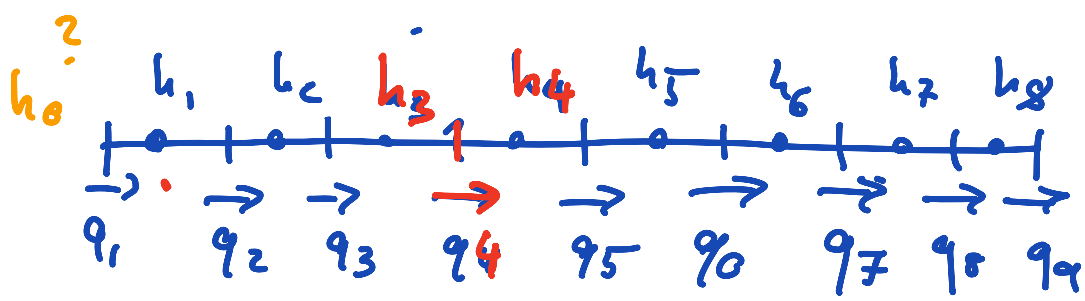
$$\underset{\text{not in }\underline{\underline{G}}}{-}\;$$

$$\begin{array}{ccccc} \underline{q} & & & \underline{\underline{G}} & \underline{h} \\ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \vdots \end{bmatrix} & = & -\dfrac{1}{\Delta x} & \begin{bmatrix} 0 & & & & & & & \\ -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & -1 & 1 & & & & \\ & & & -1 & 1 & & & \\ & & & & -1 & 1 & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \\ & & & & & & & 0 \end{bmatrix} & \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \end{bmatrix} \end{array}$$

not in $\underline{\underline{G}}$

Choose to impose natural BC

i.e., no gradient/flow



$$q_4 = -\frac{h_4 - h_3}{\Delta x}$$

Relation between $\underline{\underline{D}}$ and $\underline{\underline{G}}$?

Just looking at sizes:

$$\boxed{\underline{\underline{G}} = -\underline{\underline{D}}^T} \quad \text{in the interior}$$

but at bnd's the natural BC lead to difference

(This is due to the fact that ∇ & $\nabla\cdot$ are adjoints)

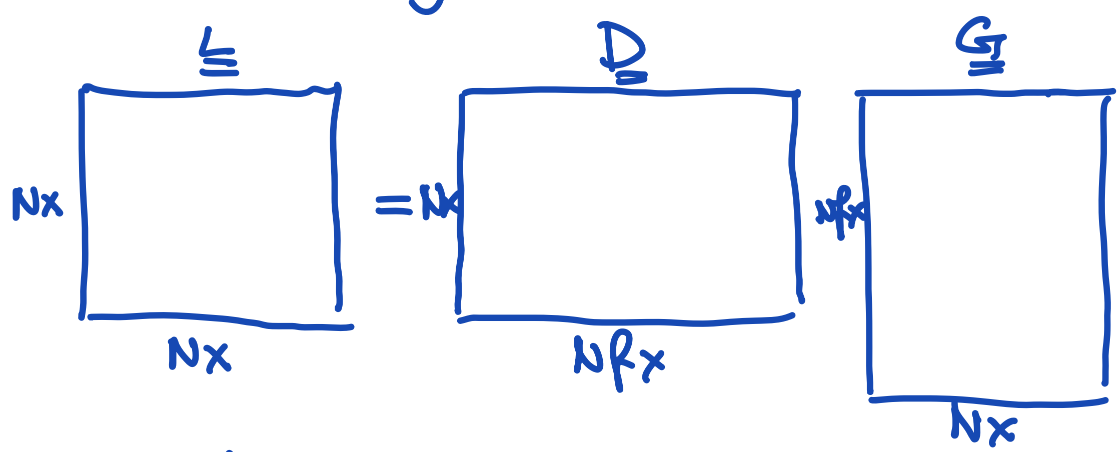
Discrete Laplacian Operator

Continuum: $\nabla\cdot\nabla = \nabla^2 \Rightarrow -\nabla^2 h = f_s$

Discrete: $\underline{\underline{D}} * \underline{\underline{G}} = \underline{\underline{L}} \Rightarrow -\underline{\underline{L}}\,\underline{h} = \underline{f}_s$

Note: Laplacian takes a scalar and returns scalar

$\Rightarrow \underline{\underline{L}}$ Nx by Nx



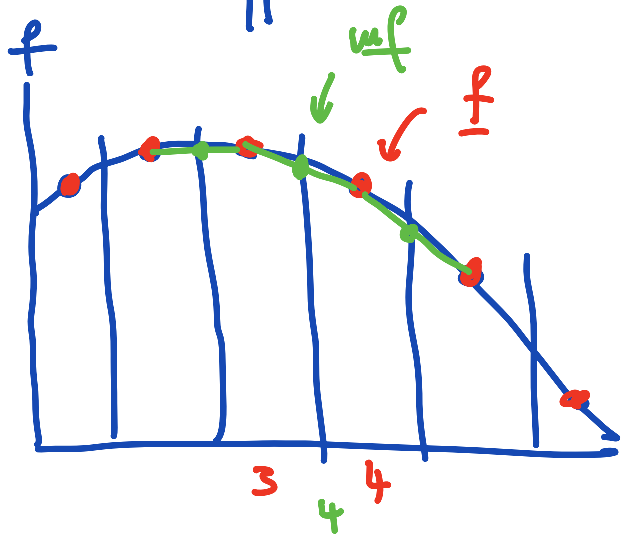
$$\underline{\underline{L}} = \frac{1}{\Delta x^2}\begin{bmatrix} -1 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & -1 & 1 \end{bmatrix}$$

Note all rows have to sum to zero because a constant can be a solution

Discrete Mean Operator

⇒ useful once we have variable coefficients $k = k(x)$

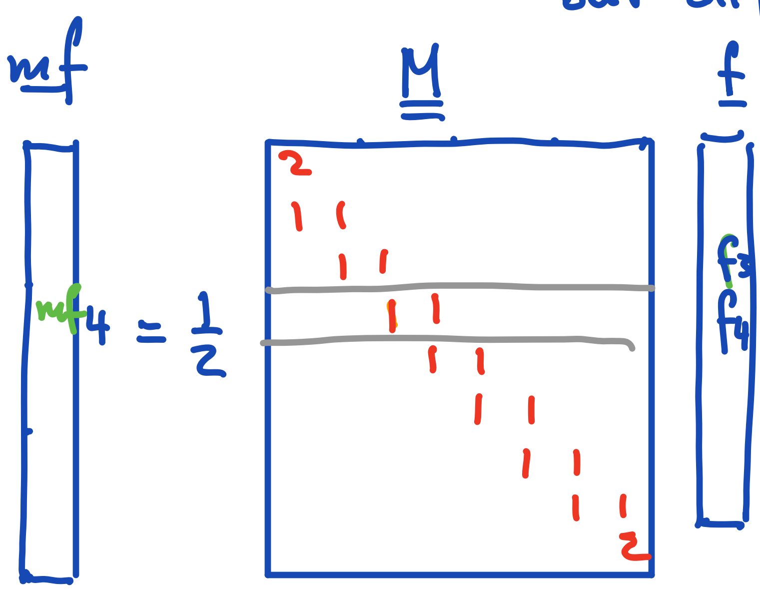
$\underline{\underline{M}}$ computes arithmetic mean of cell center values on the faces



$$\underline{mf} = \underline{\underline{M}}\;\underline{f}$$

Nfx·1   Nfx·Nx   Nx·1

⇒ $\underline{\underline{M}}$ as shape of $\underline{\underline{G}}$ but different entries



$$mf_4 = \frac{f_3 + f_4}{2}$$