Lecture 6: Shallow aquifer model

Logistics: - HW1 due next Thursday

Last time: - Discrete operators

- $\underline{\underline{G}}$ discrete gradient N_{fx} by N_x

  $\underline{h}$ in cell centers $\rightarrow$ $\underline{dh}$ ($\underline{q}$) on cell faces

- $\underline{\underline{D}}$ discrete divergence N_x by N_{fx}

  $\underline{q}$ on cell faces $\rightarrow$ $\underline{dq}$ in cell centers

  $\underline{\underline{G}} = -D^T$

- $\underline{\underline{M}}$ mean operator N_{fx} by N_x

  $\underline{h}$ in cells $\rightarrow$ $\underline{mh}$ on faces

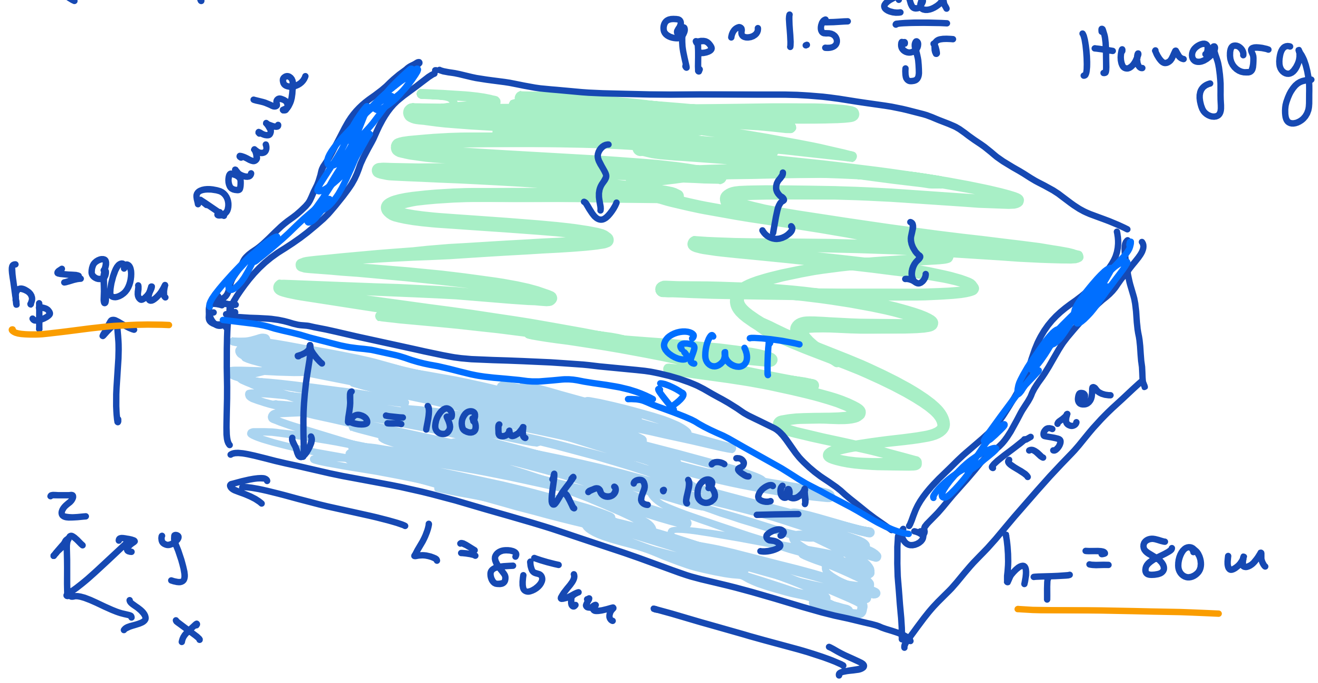
- $L = \underline{\underline{D}}\,\underline{\underline{G}}$ Laplacian N_x by N_x

Today: - Shallow aquifer model

- Dirichlet boundary conditions

Groundwater recharge between two rivers

(Sanford et al. 2001)



Aspect ratio of aquifer: $b/L = \frac{100}{85000\,m} = \frac{1}{850} \sim 0.001$

$\Rightarrow$ flow is practically horizontal ~ 1D

You can show this by scaling analysis of continuity equation $\nabla \cdot q - \cancel{f_s}^{\,0} = 0$

$$q = \begin{pmatrix} q_x \\ \cancel{q_y} \\ q_z \end{pmatrix}$$

Introduce characteristic scales:

$$x_D = \frac{x}{L} \qquad z_D = \frac{z}{b} \qquad q_{x,D} = \frac{q_x}{q_{x,c}} \qquad q_{z,D} = \frac{q_z}{q_{z,c}}$$

often some scales are not clear

substitute:

$$\nabla\cdot q = \frac{\partial q_x}{\underset{\uparrow\, x_D L}{\partial x}} + \frac{\partial q_z}{\underset{\uparrow\, z_D b}{\partial z}} = \frac{q_{x,c}}{L}\frac{\partial q_{x,D}}{\partial x_D} + \frac{q_{z,c}}{b}\frac{\partial q_{z,D}}{\partial z_D} = 0$$

$$\frac{\partial q_{x,D}}{\partial x_D} + \underbrace{\frac{q_{z,c}L}{q_{x,c}b}}_{\Pi}\frac{\partial q_{z,D}}{\partial z_D} = 0$$

Π dimensionless parameter

Set $\Pi = 1$ get relation between fluxes

$$\boxed{q_{z,c} = \frac{b}{L}\, q_{x,c}}$$

$\uparrow$ 0.001

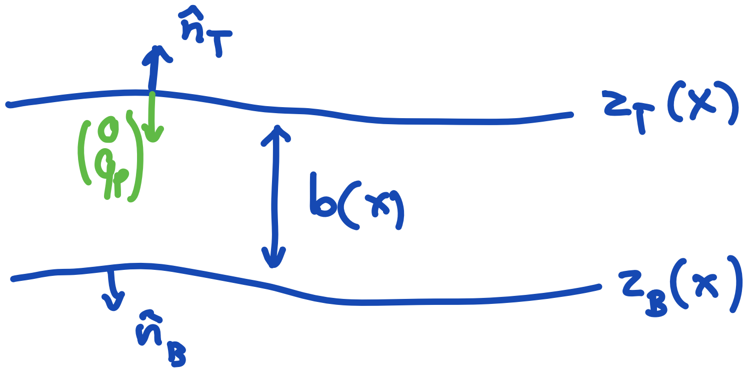
$\Rightarrow$ vertical flux is negligible

Assume $q_z = 0 \Rightarrow \frac{\partial h}{\partial z} = 0 \Rightarrow h = h(x,y)$

$$\cancel{q_z}^{\,0} = -K \cancel{\frac{\partial h}{\partial z}} \rightarrow \text{const}$$

Darcy: $q_h = \begin{pmatrix} q_x \\ q_y \end{pmatrix} = -K \underset{\substack{\uparrow \\ \text{horizontal gradient}}}{\nabla_h} h \qquad \nabla_h h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix}$

Vertical integration



$$b(x) = z_T(x) - z_B(x)$$

$$\int_{z_B(x)}^{z_T(x)} \nabla \cdot \underset{\sim}{q} \, dz = ?$$

Need to exchange integral and derivative

but z_T & z_B depend on x!

Standard Leibnitz integral rule:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,z)\,dz = \int_{a(x)}^{b(x)} \frac{df}{dx}\,dz + f(x, b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx}$$

Here we need slightly different version (Pinder & Gray)

$$\int_{z_B}^{z_T} \nabla \cdot \underset{\sim}{q}\, dz = \nabla_h \cdot \int_{z_B}^{z_T} q_h \, dz + (\hat{n}\cdot\underset{\sim}{q}\big|_{z_T} - (\hat{n}\cdot\underset{\sim}{q}\big|_{z_B}$$

since $q_h \neq q_h(z)$

$$\Rightarrow \quad \nabla_h \cdot \int_{z_B}^{z_T} q_h \, dz = \nabla_h \cdot \left(q_h \int_{z_B}^{z_T} dz \right) = \nabla_h \cdot (b\, q_h)$$

Boundary terms:

Bottom is impermeable: $(\hat{n} \cdot q)\big|_B = 0$

Top: $(\hat{n} \cdot q)\big|_{z_T} = -q_p$

Substitute: $\nabla_h \cdot (b\, q_h) - q_p = 0$

Darcy: $q_h = -K \nabla_h h$

$$\Rightarrow \quad \boxed{-\nabla_h \cdot (b K \nabla_h h) = q_p}$$

$b \leftarrow h$

2D shallow aquifer model (saturated) confined

confined

Note: z_T is water table!

$z_T = h \qquad z_B = 0 \qquad \Rightarrow \quad b = h$

In 1D we have: $-\dfrac{d}{dx}\left[b K \dfrac{dh}{dx}\right] = q_p$

<u>Simplified example problem:</u>

PDE: $-\dfrac{d}{dx}\left[bk \dfrac{dh}{dx}\right] = q_p \qquad x \in [0, L]$

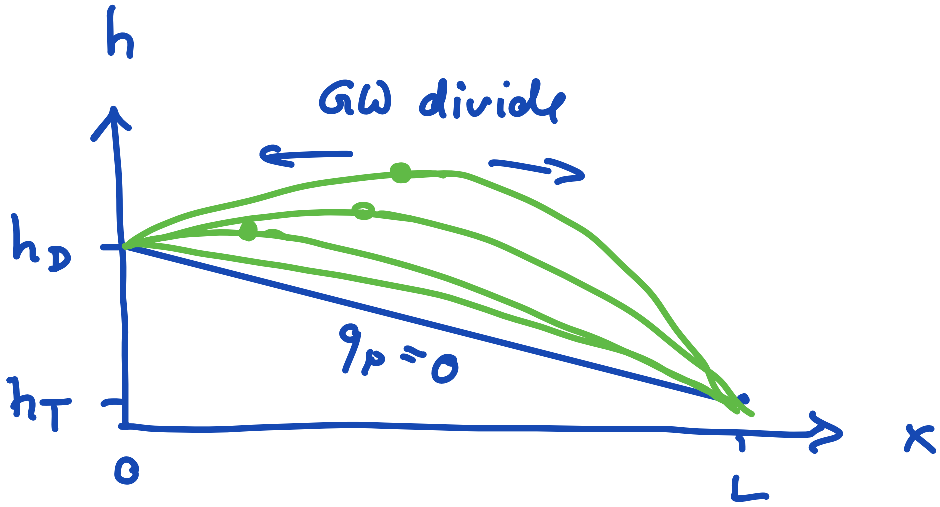
BC: $h(0) = h_D \qquad h(L) = h_T$

$b, k, q_p = \text{const.}$

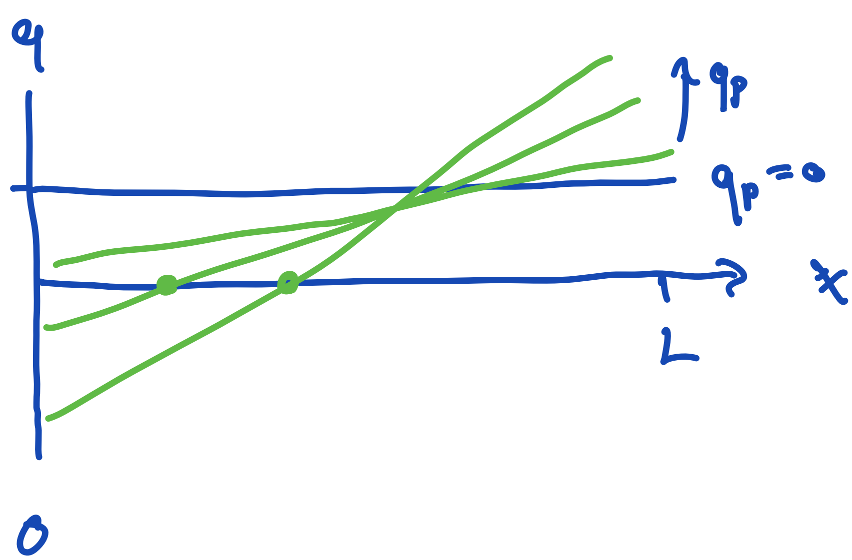
Integrate twice to obtain analytic solution:

$$\boxed{\begin{aligned} h &= h_D + \left(\frac{h_T - h_D}{L} + \frac{q_p L}{2bK}\right) x - \frac{q_p}{2bK} x^2 \\ q &= \frac{q_p}{b}\left(x - \frac{L}{2}\right) - \frac{K}{L}\left(h_T - h_D\right) \end{aligned}}$$

Sketch solution



As recharge/precipitation increases a "ground water divide" forms that separates water flowing into Danube from water flowing into Tisza river.



⇒ solve numerically

Dirichlet BC and Constraints

BC are required for problem to be well posed!

Dir BC specify h on bnd.

$\Rightarrow$ constraints on solution

$\Rightarrow$ reduces the number of unknowns

Need to learn how to eliminate constraints!

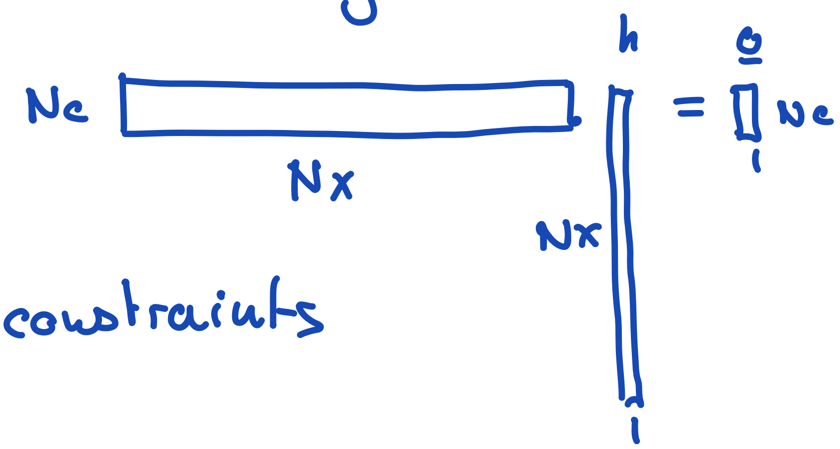
Step 1: Homogeneous Dirichlet BC's

PDE: $\quad -\nabla\cdot[bk\nabla h] = q_p = 1 \qquad x \in [0,L]$

BC: $\quad h(0) = h(L) = 0$

Write BC as linear system:

$$\underline{\underline{B}}\,\underline{h} = \underline{0}$$



Nc = number of constraints

$N_c \ll N_x$

Full discrete problem

$$\boxed{\begin{array}{ll} \text{PDE:} & \underline{\underline{L}}\,\underline{h} = \underline{f}_s \\ \text{BC:} & \underline{\underline{B}}\,\underline{h} = \underline{0} \end{array}}$$

$\underline{\underline{L}}$ Nx by Nx system matrix

$\underline{\underline{B}}$ Nc by Nx constraint matrix

Next time combine $\underline{\underline{B}}$ and $\underline{\underline{L}}$ into reduced linear system $(N_x - N_c)$ by $(N_x - N_c)$