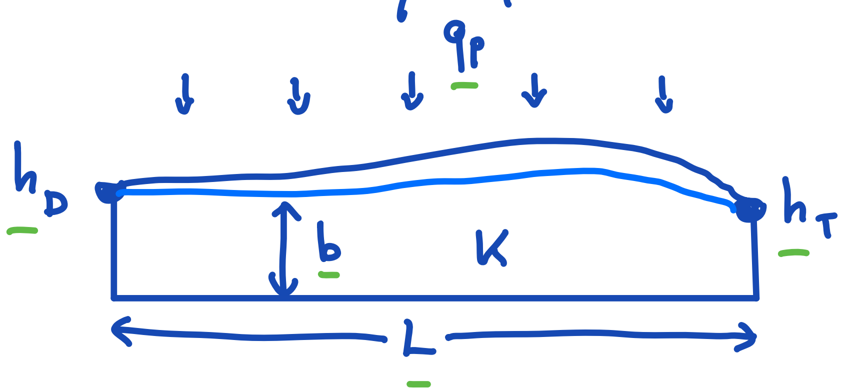
Lecture 7: Dirichlet BC's & Constraints

Logistics: - HW1 due Thursday 2/8/24 9:30 am

P1 : 8 P2 : 4

- late submission is 10% off
- Office hours : today 3-4pm JGB 4.216G
- undergrads required to attend in person

Last time: - Shallow aquifer model



- Large aspect ratio ⇒ 1D flow $L >> b$
- Vertically integrate and reduced 1D model

PDE: $-\frac{d}{dx}\left(bK\frac{dh}{dx}\right) = q_p \qquad x \in [0 \ L]$

BC: $h(0) = h_0 \qquad h(L) = h_T$

Today : - Implementation of Dirichlet BC's

- Eliminate Constraints

Dirichlet BC & Constraints

⇒ simplified homogeneous BC

PDE: $\quad -\cancel{bk}\,\nabla\cdot\nabla h = \dfrac{q_p}{bk} \qquad x \in [0, L] \qquad b, K, q_p = \text{const.}$

BC: $\quad h(0) = h(L) = 0$

Discretize PDE:

$$-\underbrace{\underline{\underline{D}}\,\underline{\underline{G}}}\,\underline{h} = \underline{f}_s$$

$$\boxed{\underline{\underline{L}}\,\underline{h} = \underline{f}_s}$$

$$\underline{f}_s = \frac{q_p}{bk}\,\text{ones}(Nx, 1)$$

Grid.Nx

bk

$$\underline{\underline{L}} = -\,\underline{\underline{D}}\,\underline{\underline{G}}$$

system matrix

Matlab note: $\quad \underline{h} = \underline{\underline{L}} \setminus \underline{f}_s$

what not to do: $\quad \underline{\underline{L}}\,\underline{h} = \underline{f}_s$

$$\cancel{\underline{\underline{L}}^{-1}\underline{\underline{L}}}^{\,\underline{\underline{I}}}\,\underline{h} = \underline{\underline{L}}^{-1}\underline{f}_s$$

$$\underline{h} = \underline{\underline{L}}^{-1}\underline{f}_s$$

mathematically correct

$$\underline{h} = \text{inv}(\underline{\underline{L}})\,\underline{f}_s$$

but bad idea computationally

Note: $\underline{\underline{L}}$ is not invertible because there are infinite solns, without BC's.

Need to write BC as a linear system

$$\underline{\underline{B}}\,\underline{h} = \underline{0}$$

$N_x = 8$

$\boxed{h_1 = 0}$ $\qquad h_8 = 0$

$N_c = 2$

$$\begin{array}{c} \\ r_1 \rightarrow \\ \\ \end{array}
\overset{\underline{\underline{B}}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}
\overset{\underline{h}}{\begin{bmatrix} h_1 \\ \vdots \\ h_8 \end{bmatrix}} = \overset{\underline{0}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\underline{r}_1 \cdot \underline{h} = 0$$

$$h_1 = 0$$

Full discrete problem:

PDE: $\underline{\underline{L}}\,\underline{h} = \underline{f}_s$

BC: $\underline{\underline{B}}\,\underline{h} = \underline{0}$

$\underline{\underline{L}}$ = N_x by N_x system matrix

B = N_c by N_x constraint mat.

Neither system has a unique solution but together they do!

$\Rightarrow$ Combine them by eliminating the constraints in $\underline{\underline{B}}$ from $\underline{\underline{L}}$.

Reduced Linear System

Constraints reduce the number of unknowns

$\Rightarrow$ solve a smaller system

unknowns : $N_x - N_c$

Reduced system $\boxed{\underline{\underline{L}}_r \, \underline{h}_r = \underline{f}_{s,r}}$

$\underline{h}_r$ is $N_x - N_c$ by 1 reduced solution vector

$\underline{f}_{s,r}$ is $N_x - N_c$ by 1 " r.h.s "

$\underline{\underline{L}}_r$ is $(N_x - N_c)$ by $(N_x - N_c)$ " system matrix

What is the relation between : $\underline{h}_r$ and $\underline{h}$

$\underline{f}_{s,r}$ and $\underline{f}_s$

$\underline{\underline{L}}_r$ and $\underline{\underline{L}}$

Projection matrix

Two vectors of different length are related by rectangular matrix

$$\underset{N_x \cdot 1}{\underline{h}} = \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \; \underset{N_x - N_c \cdot 1}{\underline{h}_r}$$

What is $\underline{\underline{N}}$?

For now just require that $\underline{\underline{N}}$ is orthonormal

$$\underline{\underline{N}} = \begin{bmatrix} | & | & | & & | \\ \underline{n}_1 & \underline{n}_2 & \underline{n}_3 & \cdots & \underline{n}_i \\ | & | & | & & | \end{bmatrix}$$

$\underline{n}_i$ are columns of $\underline{\underline{N}}$

$$\underline{n}_i \cdot \underline{n}_{j \neq i} = 0$$

$$\underline{n}_i \cdot \underline{n}_i = 1$$

It follows:

a) $$\underset{(N_x - N_c) \cdot N_x}{\underline{\underline{N}}^T} \; \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} = \underset{(N_x - N_c) \cdot (N_x - N_c)}{\underline{\underline{I}}_r}$$

b) $$\underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} \; \underset{(N_x - N_c) \cdot N_x}{\underline{\underline{N}}^T} = \underset{N_x \cdot N_x}{\underline{\underline{I}}'}$$

$\underline{\underline{I}}'$ "identity" in fullspace but with N_c zeros on the diagonal

$\underline{\underline{I}}_r$ proper identity in reduced space if $\underline{h} = \underline{\underline{N}}\,\underline{h}_r$

$$\underline{\underline{N}}^T \underline{h} = \underline{\underline{N}}^T \underline{\underline{N}}\,\underline{h}_r = \underline{\underline{I}}_r\,\underline{h}_r = h_r$$

$$\Rightarrow \underline{h}_r = \underline{\underline{N}}^T \underline{h}$$

$$\Rightarrow \boxed{\begin{aligned} \underline{h} &= \underline{\underline{N}}\,\underline{h}_r \\ \underline{h}_r &= \underline{\underline{N}}^T \underline{h} \end{aligned}}$$

$\underline{\underline{N}}$ allows us to go between full and reduced space

We say that $\underline{\underline{N}}^T$ projects $\underline{h}$ into the reduced space.

Similarly for rhs: $\underline{f}_s = \underline{\underline{N}}\,\underline{f}_{s,r}$

$\underline{f}_{s,r} = \underline{\underline{N}}^T \underline{f}_s$

How is $\underline{\underline{L}}$ projected into reduced space?

$$\underline{\underline{L}}\,\underline{h} = \underline{f}_s \qquad \underline{\underline{L}}_r\,\underline{h}_r = \underline{f}_{s,r}$$

$$\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{h} = \underline{\underline{N}}^T \underline{f}_s = \underline{f}_{s,r}$$

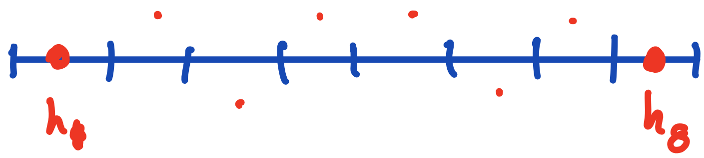
$$\underbrace{\underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}}_{\underline{\underline{L}}_r}\,\underbrace{\underline{\underline{N}}^T \underline{h}}_{\underline{h}_r} = \underline{f}_{s,r}$$

$\underline{\underline{I}}'$

$$\underline{\underline{L}}_r \, \underline{h}_r = \underline{f}_{s,r} \quad \Rightarrow \quad \boxed{\begin{aligned} \underline{\underline{L}}_r &= \underline{\underline{N}}^T \underline{\underline{L}} \, \underline{\underline{N}} \\ \underline{h}_r &= \underline{\underline{N}}^T \underline{h} \\ \underline{f}_{s,r} &= \underline{\underline{N}}^T \underline{f}_s \end{aligned}}$$

Now we just need to find $\underline{\underline{N}}$!

$\underline{\underline{N}}$ needs to contain information about boundary conditions, i.e., $\underline{\underline{B}}$



$$\underline{\underline{B}} \, \underline{h} = \underline{0}$$

We need to search for solutions in the null space of $\underline{\underline{B}}$, i.e., all solutions that satisfy $\underline{\underline{B}} \, \underline{h} = 0$

$\underline{\underline{N}}$ needs to project $\underline{h}$ into the null space of $\underline{\underline{B}}$, because if $\underline{h}$ is not $\mathcal{N}(\underline{\underline{B}})$ then it does not satisfy BC's.

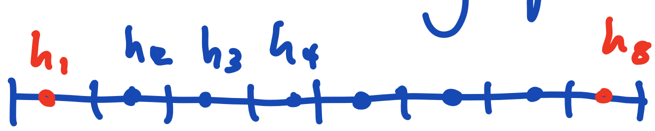
The matrix $\underline{\underline{N}}$ can be any orthonormal basis for null space of $\underline{\underline{B}}$.

In Matlab we can find null space

$$\underline{\underline{N}} = \text{null}(\underline{\underline{B}})$$

$$\underline{\underline{N}} = \text{spnull}(\underline{\underline{B}}) \qquad (\text{ download File exchange})$$

This takes long for big systems, but it turns out we can easily find basis ourselves:



$$\underline{\underline{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{N}}$$

$$\underline{\underline{I}} = \begin{array}{c} \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} h_2 \\ h_3 \\ h_4 \\ \vdots \\ \\ h_7 \end{array}\right\} \\ \\ \end{array}$$

$$\underline{\underline{N}} = \begin{bmatrix} 0 & 0 & 0 & & & & \\ 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & & 1 \\ & & & 0 & & 0 & 0 \end{bmatrix}$$

This is a orthonormal basis for the null space of $\underline{\underline{B}}$

Heterogeneous BCs

$$\underline{\underline{B}}\,\underline{h} = \underline{g} \qquad\qquad \underline{g} = \begin{bmatrix} h_D \\ h_T \end{bmatrix}$$

$\underline{\underline{B}}$ is same as before, because it specifies location of BC's

because $\underline{\underline{B}}\,\underline{h} = \underline{g}$ is linear we can decompose

$$\underline{h} = \underline{h}_0 + \underline{h}_p$$

homogeneous: $\underline{\underline{B}}\,\underline{h}_0 = \underline{0}$

heterogeneous: $\underline{\underline{B}}\,\underline{h}_p = \underline{g}$

$$\left.\begin{array}{l}\underline{\underline{B}}\,\underline{h}_0 = \underline{0}\\ \underline{\underline{B}}\,\underline{h}_p = \underline{g}\end{array}\right\} \quad \underline{\underline{B}}\,\underbrace{(\underline{h}_0 + \underline{h}_p)}_{\underline{h}} = \underline{g} + \underline{0} = \underline{g}$$

Note: $\underline{h}$ is unique

but split $\underline{h} = \underline{h}_0 + \underline{h}_p$ is not unique

but there is a simplest obvious choice

Two questions:

1) How do we find $\underline{h}_p$?

2) Given $\underline{h}_p$ how do we find associated $\underline{h}_0$?

Start with 2: Suppose we knew $\underline{h}_p$

$$\underline{\underline{L}}\,(\underline{h}_0 + \underline{h}_p) = \underline{f}_s$$

$\underline{h}_p$ is known $\Rightarrow$ rhs

$$\underline{\underline{L}}\,\underline{h}_0 = \underline{f}_s \underbrace{- \underline{\underline{L}}\,\underline{h}_p}_{\underline{f}_D} = \underline{f}_s + \underline{f}_D$$

$$\underline{f}_D = -\underline{\underline{L}}\,\underline{h}_p$$

$\Rightarrow$ reduced system: $\underline{\underline{L}}_r\,\underline{h}_r = \underline{f}_r$

$$\underline{f}_r = \underline{\underline{N}}^T(\underline{f}_s + \underline{f}_D)$$