Lecture 8: Layered Media & Effective Conductivity

Logistics: - HW 1 is due (P1 16/17, P2 14/16)

⇒ everybody needs to fully complete home work

⇒ come see me at office hours if you have problems

- HW 2 is posted today due next Th.

Last time: - Dirichlet BC

- PDE: $\underline{\underline{L}}\,\underline{h} = \underline{f}_s$ $\quad \underline{\underline{L}}$ system matrix

BC: $\underline{\underline{B}}\,\underline{h} = \cancel{\underline{g}}\ 0$ $\quad \underline{\underline{B}}$ constraint matrix

- Reduced system $\boxed{\underline{\underline{L}}_r \underline{h}_r = \underline{f}_{s,r}}$

$(Nx-Nc)\cdot(Nx-Nc)$ $\quad \underline{h}_r = \underline{\underline{N}}^T \underline{h} \qquad \underline{f}_{s,r} = \underline{\underline{N}}^T \underline{f}_s$

$\underline{\underline{L}}_r = \underline{\underline{N}}^T \underline{\underline{L}}\,\underline{\underline{N}}$

Today: - Heterogeneous BC

- Layered media

↳ heterogeneous coefficients

Finish heterogeneous BC's:

$$\underline{h} = \underline{h}_0 + \underline{h}_p \qquad \underline{\underline{B}}\,\underline{h}_0 = \underline{0} \qquad \underline{\underline{B}}\,\underline{h}_p = \underline{g}$$

$$\underline{\underline{L}}\,\underline{h} = \underline{f}_s \qquad \underline{\underline{L}}\,\underline{h}_0 \neq \underline{f}_s \qquad \underline{\underline{L}}\,\underline{h}_p \neq \underline{f}_s$$

Two questions:

1) How do we find $\underline{h}_p$?
2) Given $\underline{h}_p$, how do we find the associated $\underline{h}_0$?

Start with 2:

$$\underline{\underline{L}}(\underline{h}_0 + \underline{h}_p) = \underline{f}_s$$

known $\longrightarrow$ rhs

$$\underline{\underline{L}}\,\underline{h}_0 = \underline{f}_s - \underline{\underline{L}}\,\underline{h}_p = \underline{f}_s + \underline{f}_p$$

$$\underline{f}_p = -\underline{\underline{L}}\,\underline{h}_p$$

solve as before:
$$\underline{\underline{L}}_r\,\underline{h}_{0,r} = \underline{f}_r$$
$$\underline{f}_r = \underline{\underline{N}}^T(\underline{f}_s + \underline{f}_p)$$

$$\Rightarrow \quad \underline{h} = \underline{h}_0 + \underline{h}_p$$

Q1: How do we find $\underline{h}_p$?

$\underline{h}_p$ is not unique: $\underline{\underline{B}}\,\underline{h}_p = \underline{g}$

Simplest solution: $\underline{h}_p = \begin{bmatrix} h_D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_T \end{bmatrix}$

In higher dimensions we want to place entries automatically:

$\underline{h}_{pr} = \underline{\underline{B}}\,\underline{h}_p \quad \longrightarrow \quad \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$

$\Rightarrow \quad \underline{\underline{B}}\,\underline{h}_p = \underline{g}$

$\underbrace{\underline{\underline{B}}}_{N_c \cdot N_x}\,\underbrace{\underline{\underline{B}}^T}_{N_x \cdot N_c} \underline{h}_{pr} = \underbrace{\underline{g}}_{N_c \cdot 1} \qquad \underbrace{BB^T}_{N_c \cdot N_c}$

Summary of BC implementation

Next HW write two functions

1) $[\underline{\underline{B}}, \underline{\underline{N}}, f_n]$=build_bnd(BC

2) h = solve_lbvp(   )

Inside solve-lbup

1) Find h_p

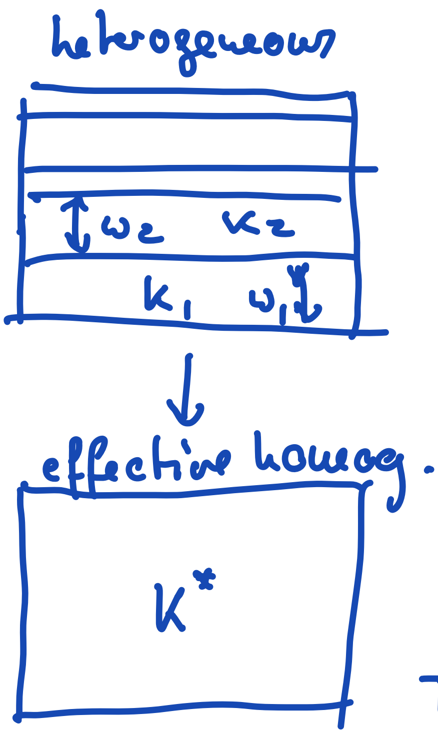
$$\underline{\underline{B}}\,\underline{\underline{B}}^T \underline{h}_{pr} = \underline{g} \quad \Rightarrow \quad \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$$

2) Find associated hom. solution

$$\underline{\underline{L}}_r \underline{h}_{0r} = \underline{f}_r \quad \Rightarrow \quad \underline{h}_0 = \underline{\underline{N}}\, \underline{h}_{0r}$$

3) Full solution: $\underline{h} = \underline{h}_0 + \underline{h}_r$

Effective Conductivity of Layered Systems



Stack of N layers of thickness w_i and conductivity K_i

How to compute K^* from w_i K_i?

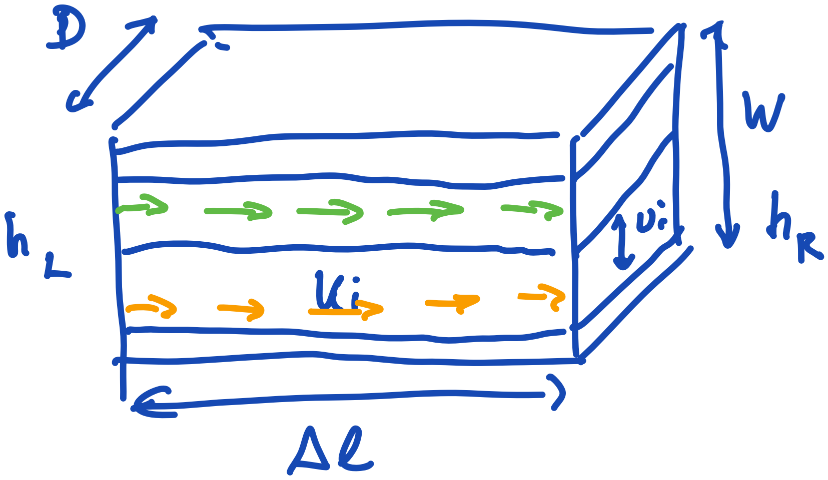
Two limiting cases:

1) Flow along layers (parallel)
2) Flow across layers (perp.)

1) Flow along layers

Head gradient left to right

no flow BC on all other bnds.



$$W = \sum_{i=1}^{N} w_i$$

$$\Delta h = h_R - h_L < 0$$

Darcy in i-th layer: $Q_i = -D w_i K_i \frac{\Delta h}{\Delta L}$

Darcy for whole stack: $Q = -D W K^*_{\parallel} \frac{\Delta h}{\Delta L}$

what is $K^*_{\parallel}$ given w_i and K_i?

$$Q = \sum_{i=1}^{N} Q_i = +\sum \cancel{D} w_i k_i \cancel{\frac{\Delta h}{\Delta L}} = +\cancel{D} W K^*_{\parallel} \cancel{\frac{\Delta h}{\Delta L}}$$

solve for $K^*_{\parallel}$

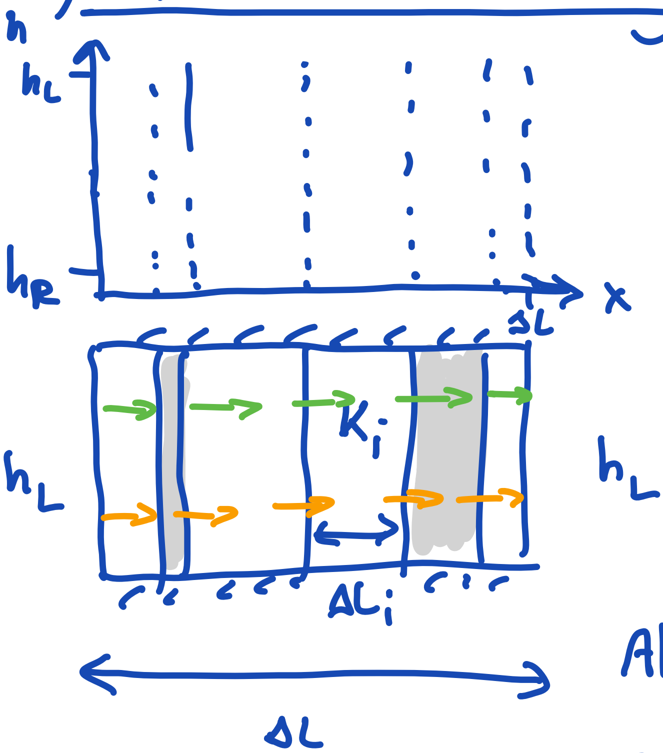
$$\boxed{K^*_{\parallel} = \sum_{i=1}^{N} \frac{w_i}{W} K_i}$$

weighted arithmetic average

⇒ high-K layers dominate behavior

2) Flow across layers

D·W



Cross-sectional area is const

⇒ consider q_i in each layer

i-th layer: $q_i = -K_i \frac{\Delta h_i}{\Delta l_i}$

whole stack: $q = -K_\perp^* \frac{\Delta h}{\Delta l}$

$\Delta h = \sum_{i=1}^{N} \Delta h_i$ $\qquad$ $\Delta l = \sum_{i=1}^{N} \Delta l_i$

All layers experience same q

$q_i = q$

$\Delta h = \sum \Delta h_i =$

$q_i = -k_i \frac{\Delta h_i}{\Delta l_i}$

$\Delta h_i = -\frac{q\, \Delta L_i}{K_i}$

Solve for $K_\perp^*$:

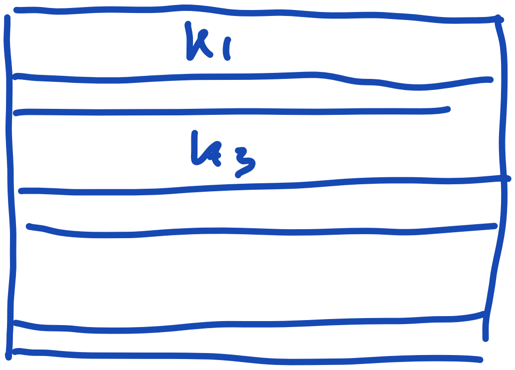
$$K_\perp^* = -\frac{q\,\Delta l}{\Delta h} = -\frac{q\,\Delta L}{\sum \Delta h_i} = -\frac{\cancel{q}\,\Delta L}{-\sum \frac{\cancel{q}\,\Delta l_i}{K_i}}$$

$$\boxed{K_\perp^* = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i/\Delta L}{K_i}}}$$

harmonic average

⇒ low k layers dominate!

fine scale

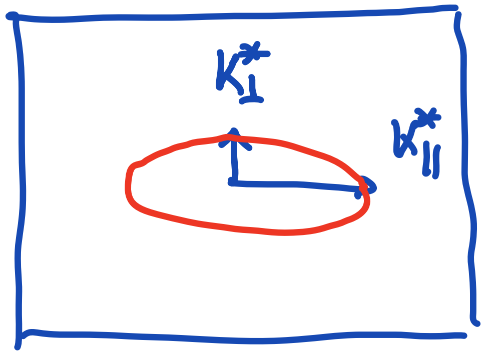


heterogeneous

$K(\underline{x})$

isotropic

coarse scale



homogeneous

anisotropic

$$\underline{\underline{K}}^* = \begin{pmatrix} K^*_\parallel & 0 & 0 \\ 0 & K^*_\parallel & 0 \\ 0 & 0 & K^*_\perp \end{pmatrix}$$

$\Rightarrow \underline{\underline{K}}$ is tensor/matrix