# Lecture 9: Heterogeneous Coefficients

Logistics: - HW2 due Thursday

P1 (8/17)  P2 (6/17)  P3 (6/17)

⇒ come to office hours if you have problems

- HW1 please complete P2! ⇒ until next Thurs!

Last time: - Layered media

- Upscaling / effective properties

- Flow along & across layers

$$K_{\parallel}^* = \sum_{i=1}^{N} \frac{w_i}{W} K_i$$ arithmetic average → high K

anisotropy
property depends on direction

$$K_{\perp}^* = \frac{1}{\sum_{i=1}^{N} \frac{\Delta l_i/\Delta L}{K_i}}$$ harmonic average → low K

Today: - Heterogeneous coefficients
- Radial coordinates

# Variable Coefficients

Heterogeneity is defining characteristic of natural porous medium.

$\Rightarrow$ treating heterogeneity is key

Continuous eqn.: $-\nabla \cdot [K(\underline{x}) \overset{h(x)}{\nabla h}] = f_s$

Discrete eqn: $-\underline{\underline{D}} * [\underset{\text{Kmean}}{\underline{\underline{Kd}}} * \underset{dh}{\underline{\underline{G}}\,\underline{h}}] = \underline{f_s}$

What is size of $\underline{\underline{Kd}}$ matrix?

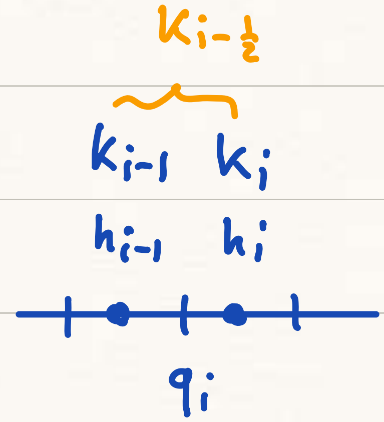
| $\underline{\underline{D}}$ | $\underline{\underline{Kd}}$ | $\underline{\underline{G}}$ |
| --- | --- | --- |
| $N_x \cdot (N_x+1)$ | $(N_x+1) \cdot (N_x+1)$ | $(N_x+1) \cdot N_x$ |

$\Rightarrow$ $\underline{\underline{Kd}}$ is a $N_x+1$ by $N_x+1$ matrix associated with the faces!

Basic problem is K's are associated with cells!

$\Rightarrow$ average K's from cells to the faces

# Entries into $\underline{\underline{Kd}}$ matrix ?




Darcy's law: $q = -K \nabla h$

$$q = -\underline{\underline{Kd}} * \underline{\underline{G}} * \underline{h}$$

$$q_i = -K_{i-\frac{1}{2}} \frac{h_i - h_{i-1}}{\Delta x}$$

$$\underline{\underline{G}}\,\underline{h} = \underline{dh}$$

where $K_{i-\frac{1}{2}}$ is average of $K_{i-1}$ and $K_i$

$$q = -\underline{Kmean} \,.\!*\, \underline{dh} \qquad \text{(element wise)}$$

$\uparrow$

$\underline{Kmean}$ is $Nx+1 \cdot 1$ vector containing averages

we can also write ~~k~~ as product between diagonal $\underline{\underline{Kd}}$ and ~~G~~ $\underline{dh}$

$$q = -\underline{\underline{Kd}}\,\underline{dh} = -\underline{\underline{Kd}} * \underline{\underline{G}} * h$$

$$\underline{\underline{Kd}} = \begin{bmatrix} \ddots & & \\ & Kmean & \\ & & \ddots \end{bmatrix} \qquad \underline{\underline{Kd}} * \underline{\underline{dh}} = \underline{Kmean} \,.\!*\, \underline{dh}$$

$\Rightarrow$ compute $\underline{Kmean}$ and place on diagonal of $\underline{\underline{Kd}}$

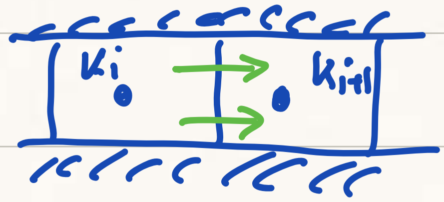
How to compute $\underline{Kmean}$?

We already have $\underline{\underline{M}}$ mean operator:

$$\underline{Kmean} = \underline{\underline{M}}\ \underline{K}$$

$\uparrow$ $N_x+1 \cdot 1$ vector of means of faces

$\underline{K}$: $N_x \cdot 1$ vector of cell conductivities




flow from one cell to another is flow across layers!

$\Rightarrow$ harmonic average

The appropriate average will depend on problem:

1) Heterogeneous conductivity

$\Rightarrow$ harmonic mean because it is flow across layers

Note: Don't need average over all cells, just neighboring cells.

$\Delta l = \Delta x \qquad \Delta l_i = \Delta l_{i-1} = \frac{\Delta x}{2}$

$$\boxed{K_{i-\frac{1}{2}} = \frac{2}{\frac{1}{K_{i-1}} + \frac{1}{K_i}}}$$

2) Non-linear conductivity: $K = K(h)$

Examples: - compressible flow (gas)

- ~~....~~ unsaturated flow $\Rightarrow$ later

Since $h$ is smooth $K(h)$ is also smooth

$\Rightarrow$ arithmetic mean may be appropriate

In these cases we have two options:

I) Evaluate then average

$$\boxed{K_{i-\frac{1}{2}} = \frac{K(h_{i-1}) + K(h_i)}{2}}$$

II) Average then evaluate

$$\boxed{K_{i-\frac{1}{2}} = K\left(\frac{h_{i-1} + h_i}{2}\right)}$$

Power-law average

Arithmetic and harmonic means are special cases of powerlaw average

$$\boxed{K_p = \left(\frac{1}{2}\left(K_{i-1}^p + K_i^p\right)\right)^{\frac{1}{p}}}$$

$p=1$ is arithmetic

$p=-1$ is harmonic

# Implementation of comp_mean.m :

HW3

Utilize $\underline{\underline{M}}$ operator from build_ops.m .

$\underline{K}$ is $Nx \cdot 1$ vector of cell conductivities

arithmetic mean ($p=1$) : $\underline{Kmean} = \underline{\underline{M}} * \underline{K}$

harmonic mean ($p=-1$): $\underline{Kmean} = 1./(\underline{\underline{M}} * (1./\underline{K}))$

or general power-law mean:

$$\boxed{\underline{Kmean} = (\underline{\underline{M}} * \underline{K}.\hat{\ } p).\hat{\ }(1/p)} \qquad p \neq 0$$

Then we place it on diagonal

$$\underline{\underline{Kd}} = \text{spdiags}(\underline{Kmean}, 0, Nf, Nf)$$