Gravity driven infiltration

Richards Equation:

$$\frac{\partial \theta}{\partial t} - \nabla \cdot \left[D(\theta) \nabla \theta - K(\theta) \hat{z} \right] = 0$$

In last few lectures → horizontal flow

$$\Rightarrow \frac{\partial \theta}{\partial t} - \nabla \cdot \left[D(\theta) \nabla \theta \right] = 0$$

non-linear diffusion equation

⇒ capillary suction of water into the soil

Today opposite limit: → gravity driven vertical flow

$$\Rightarrow \frac{\partial \theta}{\partial t} + \underbrace{\nabla \cdot \left(K(\theta) \hat{z} \right)}_{\frac{\partial}{\partial z}(K(\theta))} = 0$$

$$\boxed{\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z}\left(K(\theta) \right) = 0}$$

Advection equation (non-linear)

Brooks-Corey:

$$K(\theta) = K_s \left(\frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^{3 + \frac{2}{\lambda}} = K_s \, s^{3 + \frac{2}{\lambda}}$$

Rewrite

$$\Rightarrow \quad \frac{\partial \theta}{\partial t} + \frac{dK}{d\theta}\frac{\partial \theta}{\partial z} = 0$$

introduce velocity $v(\theta) = \frac{dK}{d\theta}$

$$\boxed{\frac{\partial \theta}{\partial t} + v(\theta)\frac{\partial \theta}{\partial z} = 0}$$

$v(s)$ is velocity at which saturation propagates!
(This can be different from fluid velocity!)

Expression for Brooks-Corey:

$$\frac{dK}{d\theta} = K_s\left(3+\frac{2}{\lambda}\right) s(\theta)^{2+\frac{2}{\lambda}}$$

$$\Rightarrow \quad v(\theta) = K_s\left(3+\frac{2}{\lambda}\right) s(\theta)^{2+\frac{2}{\lambda}}$$

Example: $\lambda = 2$

$$v(\theta) = 4K_s\, s(\theta)^3$$
$$= 4K_s\left(\frac{\theta-\theta_r}{\theta_s-\theta_r}\right)^3$$

Short Review of linear advection equation

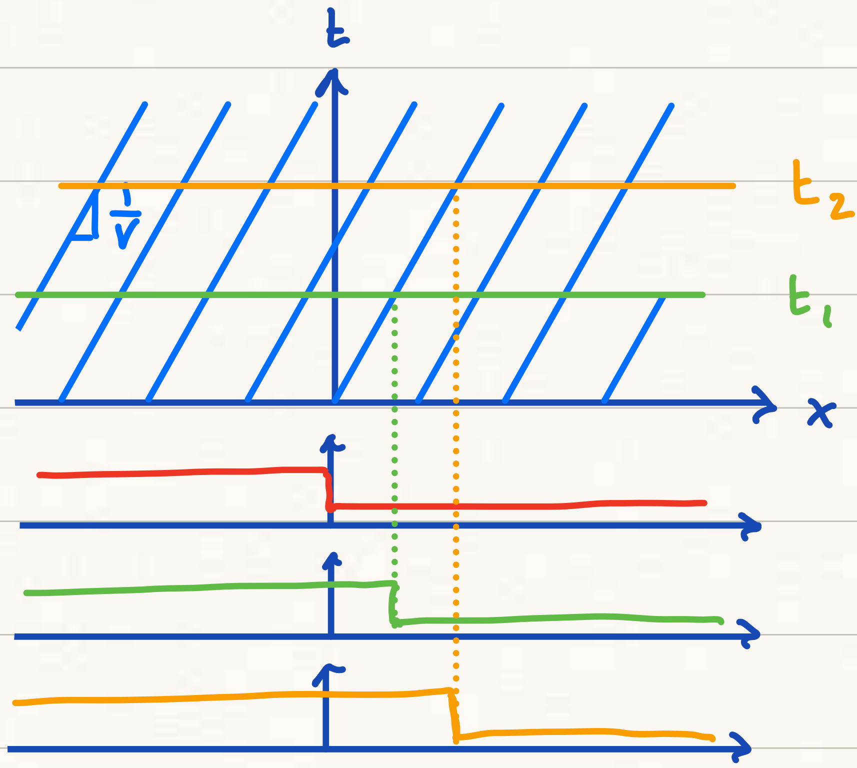
In lecture 19

$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = 0$$

$v =$ constant

Method of Char.:

$\Rightarrow$ solu is constant along lines with slope $\frac{1}{v}$ (charac.)



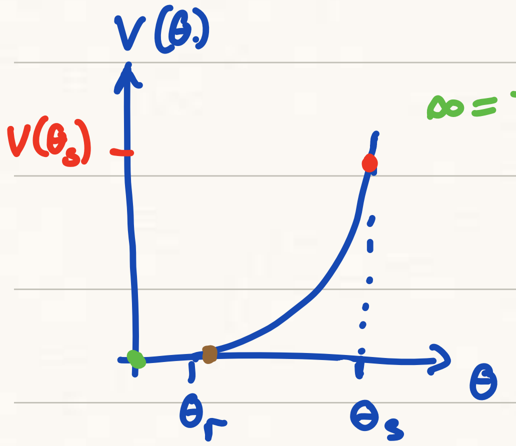
$\Rightarrow$ initial shape is translated without change in shape

Non-linear advection will change shape!

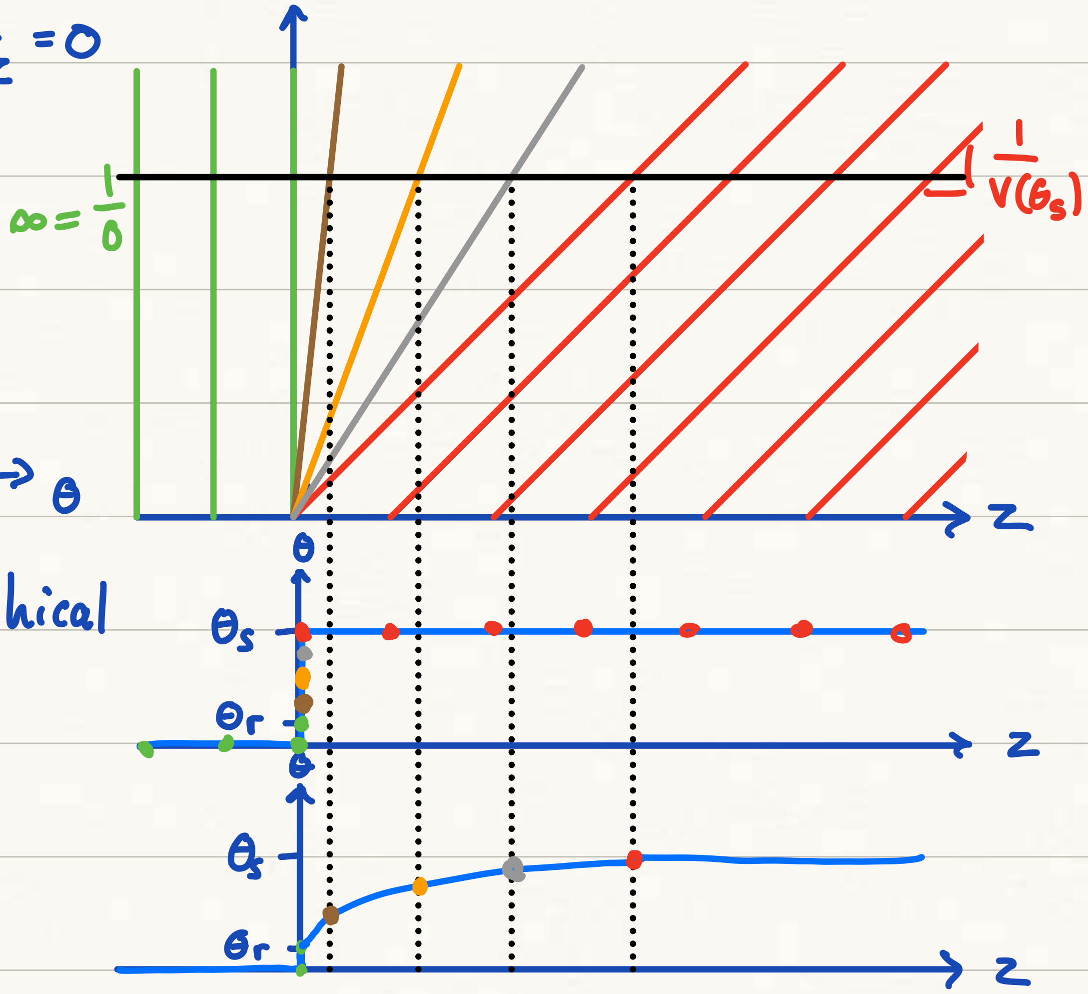
Drainage of wet soil column (drying front)

Sisson et al (1980) Soil Science of America J. 47, 3-8

$$\frac{\partial \theta}{\partial t} + v(\theta)\frac{\partial \theta}{\partial z} = 0$$



start with graphical construction



Soil is initially saturated $\theta = \theta_s$

for $x < 0$ $\quad \theta = 0$; think of this as boundary condition

no more water / rain has stopped

$\Rightarrow$ Step spreads out with time because the water at higher saturation travels faster.

"spreading wave" or "rarefaction"

We can derive variation of θ along spreading wave from char. equation as follows.

$$x - x_0 = v(\theta)\,(t - t_0) \qquad t_0 = x_0 = 0$$

$$x = 4K_s\, s^3 t$$

solve for s: $s^3 = \frac{x}{4K_s t}$

$$s = \sqrt[3]{\frac{x}{4K_s t}} \qquad s = \frac{\theta - \theta_r}{\Delta\theta} \qquad \Delta\theta = \theta_s - \theta_r$$

water content along spreading wave:

$$\theta(x,t) = \theta_r + (\theta_s - \theta_r)\sqrt[3]{\frac{x}{4K_s t}}$$

remember that this is for $\lambda = 2$ in Brooks-Corey relative hyd. conductivity.

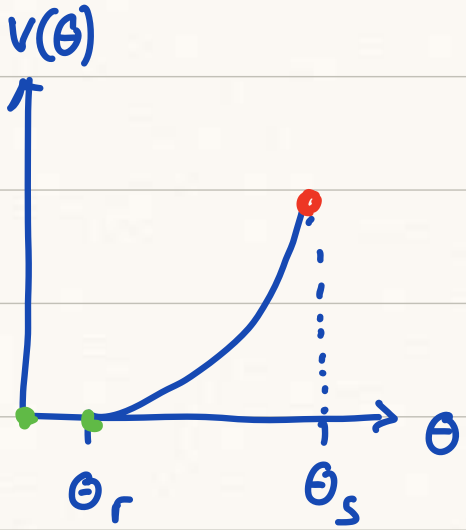
$\Rightarrow$ Shape changes with λ !

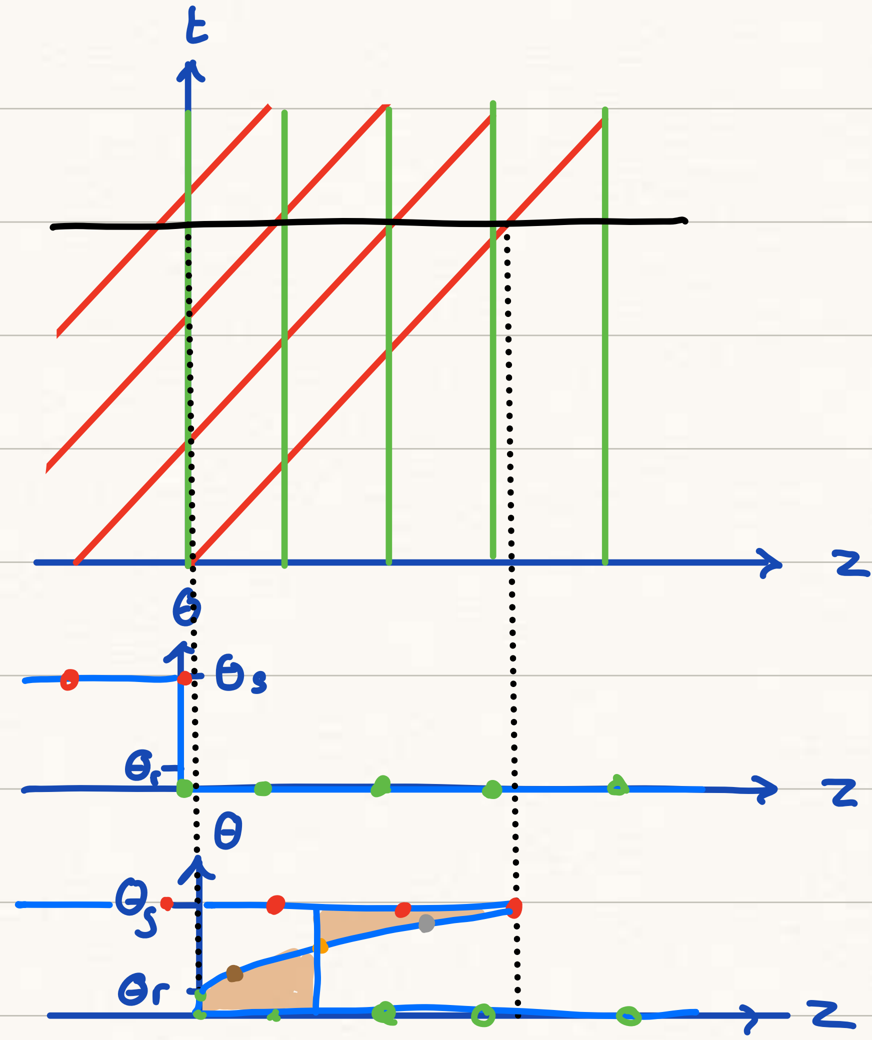
Note soln. is self-similar in $\eta = \frac{x}{t}$!

$$\theta(\eta) = \theta_r + (\theta_s - \theta_r)\sqrt[3]{\frac{\eta}{4K_s}}$$

Wetting front in initially dry soil

Now we reverse situation





Characterisics cross

⇒ solution is multivalued

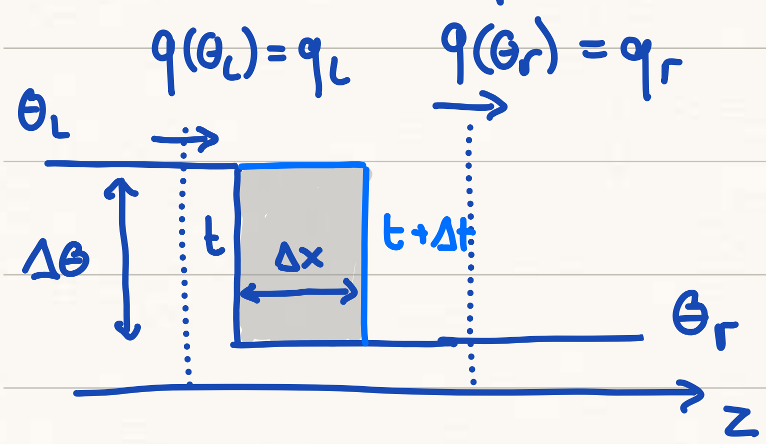
Physically not possible

⇒ introduce a shock

mass conserving step.

grapically the two shaded areas have to be the same

Derivation of shock speed



$$\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z}\Big(\underbrace{K(\theta)}_{q(\theta)}\Big) = 0$$

gravity flux

From mass/volume balance

$$(\theta_L - \theta_R)\Delta x = (q_L - q_r)\,\Delta t$$

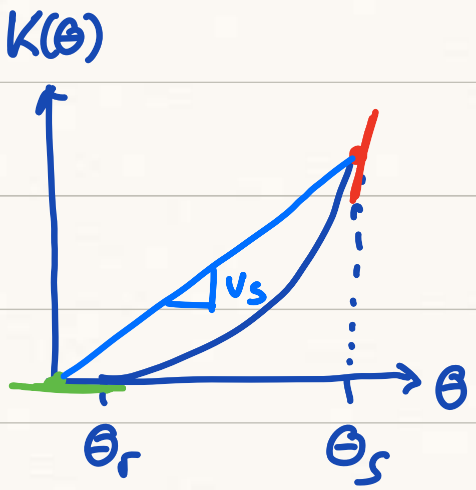
$$\frac{\Delta x}{\Delta t} = \frac{q_r - q_L}{\theta_r - \theta_L} = V_s$$

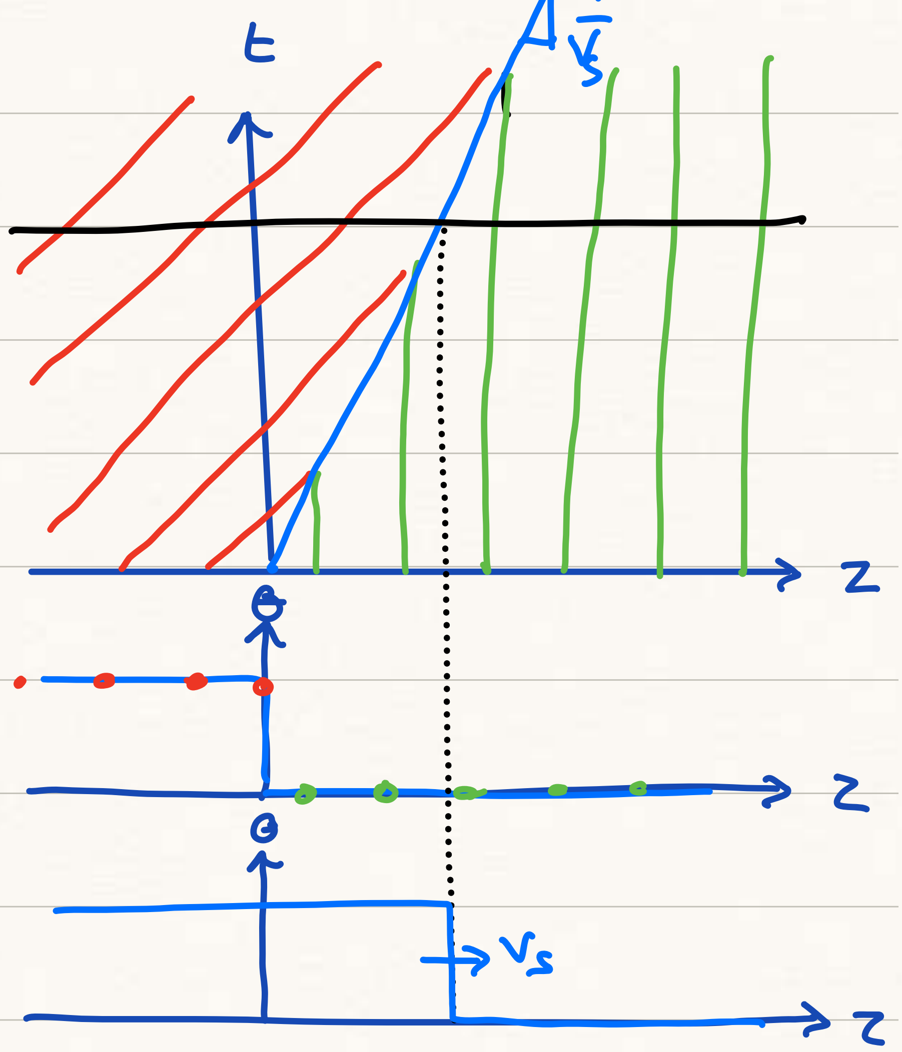
Jump condition : $\boxed{V_s = \frac{[q]}{[\theta]}}$

bracket [ ] difference between right & left states

Redo graphical construction

Notice K not v!



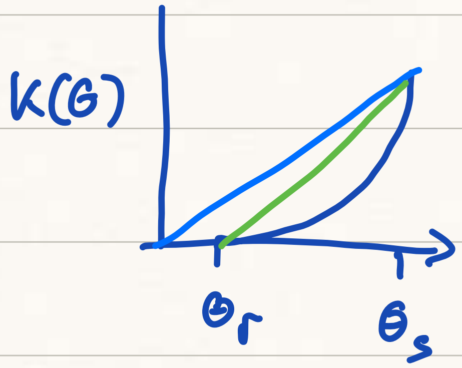


Solution is simple step moving with velocity v_s.

$\Rightarrow$ from jump condition shock speed is the cord on the flux, i.e. $K(\theta)$.

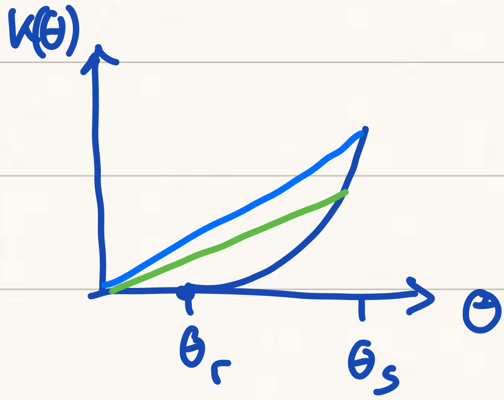
Useful because we can determine quatitative changes grapically

Q: What happens to speed of wetting front if soil in at $\theta = \theta_r$?



A: Wetting front accelerates.

Q: What happens if wetting front is only partially saturated? $\theta_i < \theta_s$



A: Wetting front slows down.

How is the wetting front different from a concentration front?

A shock is different from a concentration front because it is self-sharpening.

An initially wide front becomes increasingly narrow until a step emerges!