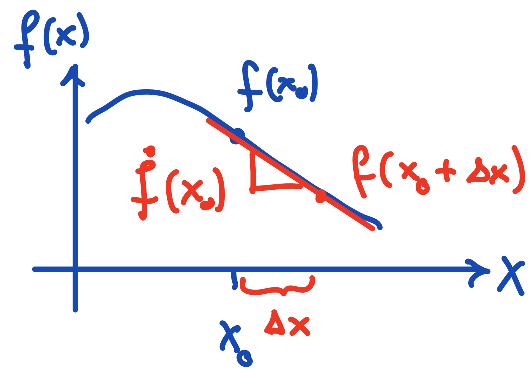
Introduction to finite differences

In calculus we define the derivative of a function as:

$$\dot{f}(x_0) = \left.\frac{df}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$



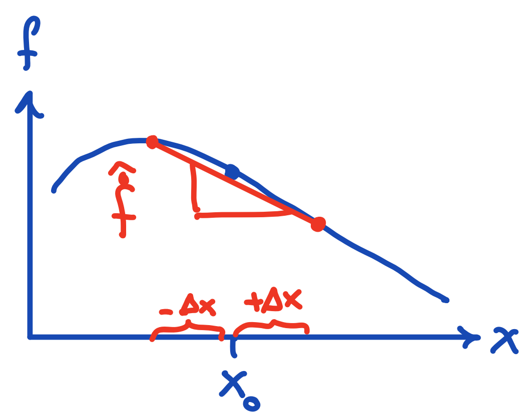
In finite difference approximation:

$$\hat{\dot{f}}(x_0) = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + O(\Delta x)$$

In numerical methods class you show that this one-sided approximation is first-order accurate, i.e. error decreases as $\frac{1}{\Delta x}$

Central finite difference

$$\hat{\dot{f}}(x_0) = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} + O(\Delta x^2)$$



$\Rightarrow$ second-order accurate

i.e., the error $\sim \frac{1}{\Delta x^2}$

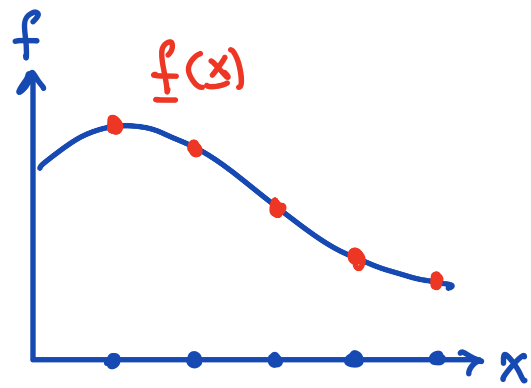
$\Rightarrow$ our go to approximation

Differentiation Matrix

The derivative is a <u>linear</u> differential operator, it takes a function and returns a different function

$$\dot{f}(x) = \underset{\substack{\uparrow \\ \text{derivative oparator}}}{\mathcal{D}}(f(x))$$

The discrete equivalent of a function is a vector, $\underline{f} = f(\underline{x})$. Similarly we can define vector $\underline{df} = \dot{f}(\underline{x})$



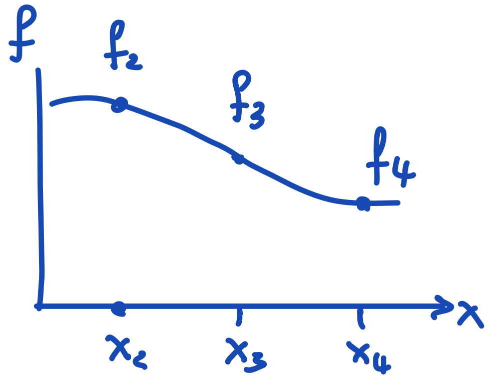
What is the discrete equivalent of $\mathcal{D}$?

$$\underline{df} = \boxed{\underline{\underline{D}}}\ \underline{f}$$

has to be a <u>matrix</u>, because it is linear and relates two vectors!

⇒ Differentiation matrix

$$\underline{df} = \begin{bmatrix} \\ df_3 \\ \\ \end{bmatrix} = \frac{1}{2\Delta x} \overset{\underline{\underline{D}}}{\begin{bmatrix} & 1 & & & & & \\ -1 & & 1 & & & & \\ & -1 & & 1 & & & \\ & & -1 & & 1 & & \\ & & & -1 & & 1 & \\ & & & & -1 & & 1 \\ & & & & & -1 & \end{bmatrix}} \begin{bmatrix} \\ f_2 \\ \\ f_4 \\ \\ \end{bmatrix} \underline{f}$$



$\Rightarrow$ $\underline{\underline{D}}$ has a simple bi-diagonal structure

Note: Boundaries require different treatment

What about 2^{nd} derivatives? $\frac{d^2 f}{dx^2}$

$$\underline{ddf} = \underline{\underline{D}}\,\underline{df} = \underline{\underline{D}}\,\underline{\underline{D}}\,\underline{f} = \underline{\underline{D}}^2\,\underline{f}$$