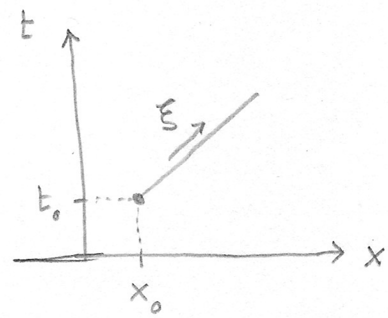
Solution to the advection equation

Analytic solution by method of characteristics

Consider tracer transport along a 1D column with constant ϕ, k

PDE: $\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = 0$ $\quad x \in \mathbb{R} \quad t \in [0,T]$

IC: $c(x,0) = c_0(x)$



Idea: Find a characteristic curve/coord., ξ, along which PDE reduces to an ODE.

$$c(x,t) = c(x(\xi), t(\xi)) = c(\xi)$$

Total change of concentration along the characteristic

$$\frac{dc}{d\xi} = \frac{\partial c}{\partial t}\frac{dt}{d\xi} + \frac{\partial c}{\partial x}\frac{dx}{d\xi}$$

PDE: $\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x}$

$\xrightarrow{\text{comparing}}$

1) $\frac{dc}{d\xi} = 0$

2) $\frac{dt}{d\xi} = 1$

3) $\frac{dx}{d\xi} = v$

(2) and 3) give) $\frac{dx}{dt} = v$

Solve eqn for characteristic: $x - x_0 = v(t - t_0)$

At the initial condition $c(x = x_0, t = t_0) = c_0(x_0)$

Substitute characteristic eqn into IC: $x_0 = x - v(t - t_0)$

Analytic solution: $\boxed{c(x,t) = c_0(x - v(t - t_0))}$

usually $t_0 = 0$ so that $c(x,t) = c_0(\underbrace{x - vt}_{\text{travelling wave coord.}})$

Definition:
A wave is a signal/disturbance/variation moving through a medium with a recognizable speed of propagation.