Time integration

Time dependent linear PDE:

$$S_s \frac{\partial h}{\partial t} \underbrace{- \nabla \cdot [K \nabla h]}_{\underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}} = f_s$$

$\Rightarrow$ just need to discretize time derivative

$$\frac{\partial h}{\partial t} = \frac{\underline{h}^{n+1} - \underline{h}^{n}}{\Delta t} \qquad \text{simple finite difference} \qquad \Delta t = t^{n+1} - t^{n}$$

substitute into eqn

$$\underline{\underline{M}} \, (\underline{h}^{n+1} - \underline{h}^{n}) - \Delta t \, \underline{\underline{L}} \, \underline{h} = \Delta t \, f_s \qquad \underline{\underline{M}} = S_s \underline{\underline{I}}$$

"mass matrix"

Theta method

We need to decide the time at which the term $\underline{\underline{L}}\underline{h}$ is evaluated: $\underline{h}^{\theta} = \theta \underline{h}^{n} + (1-\theta) \underline{h}^{n+1}$

$$\Rightarrow \quad \underline{\underline{M}} \, (\underline{h}^{n+1} - \underline{h}^{n}) + \Delta t \, \underline{\underline{L}} \, (\theta \underline{h}^{n} + (1-\theta) \underline{h}^{n+1}) = \Delta t \, \underline{f_s}$$

Collect the unknowns $\underline{h}^{n+1}$ on l.h.s.

$$\underbrace{\left[\underline{\underline{M}} + \Delta t \, (1-\theta) \, \underline{\underline{L}}\right]}_{\underline{\underline{IM}}} \underline{h}^{n+1} = \Delta t \, \underline{f_s} + \underbrace{\left[\underline{\underline{M}} - \Delta t \, \theta \, \underline{\underline{L}}\right]}_{\underline{\underline{EX}}} \underline{h}^{n}$$

Linear system for a time step: $\boxed{\underline{\underline{IM}}\ \underline{h}^{n+1} = \Delta t\, \underline{f_s} + \underline{\underline{EX}}\ \underline{h}^{n}}$

Implicit matrix: $\underline{\underline{IM}} = \underline{\underline{M}} + \Delta t\,(1-\theta)\,\underline{\underline{L}}$

Explicit matrix: $\underline{\underline{EX}} = \underline{\underline{M}} - \Delta t\,\theta\,\underline{\underline{L}}$

Properties of Theta method

For $\theta = 1$: Forward Euler Method

$\underline{\underline{IM}} = \underline{\underline{M}}$ (diagonal)

$\Rightarrow \underline{h}^{n+1} = \underline{\underline{M}}^{-1}\left(\Delta t\, \underline{f_s} + \underline{\underline{EX}}\ \underline{h}^{n}\right)$

- explicit method
- only matrix vector multiply (cheap)
- conditionally stable $\Delta t \le \frac{\Delta x^2}{2 D_{hyd}}$
- first order accurate

For $\theta = 0$: Backward Euler Method

$$\underline{\underline{EX}} = \underline{\underline{M}}$$

$$\underline{\underline{IM}}\ \underline{h}^{n+1} = \underline{f_s} + \underline{\underline{EX}}\ \underline{h}^{n}$$

- implicit method
- solve linear system at <u>every</u> timestep
- unconditionally stabe
- first order accurate

For $\theta = \frac{1}{2}$: Crank-Nicholson Method

$$\underline{\underline{IM}}\ \underline{h}^{n+1} = \Delta t\, \underline{f_s} + \underline{\underline{EX}}\ \underline{h}^{n}$$

- implicit method
- solve linear system
- unconditionally stable (but has oscillation limit)
- second-order accurate