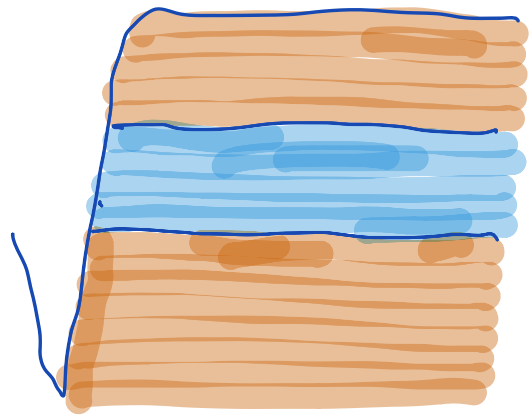
Example: Aquifer draining into Valles Marineris

Consider a linear confined aquifer draining into a suddenly created crack.

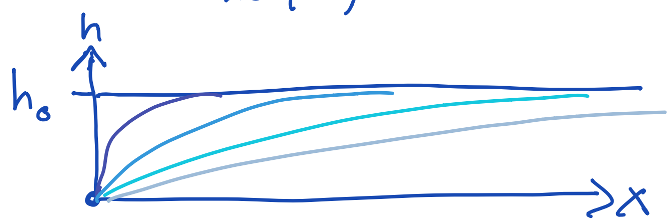


We have the following model problem:

PDE: $\frac{\partial h}{\partial t} - D \nabla^2 h = 0 \qquad x \in [0, \infty)$

BC: $h(0,t) = 0$

IC: $h(x,0) = h_0$



$D = \frac{K}{S_s}$ hydraulic diffusivity $\left[\frac{L^2}{T}\right]$

Q: How fast does the head front propagate?

Scaling the problem

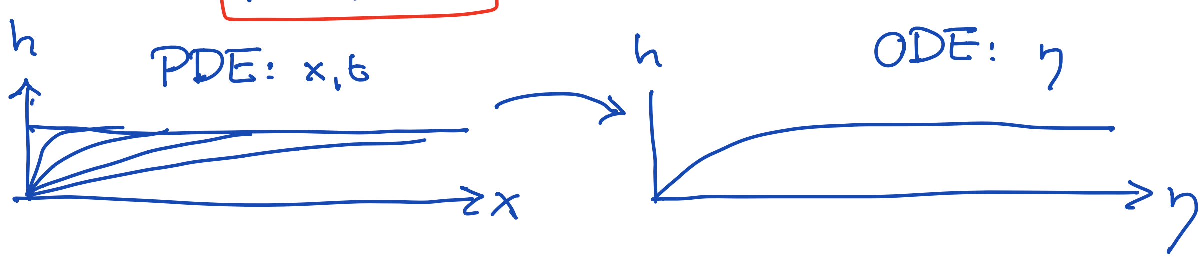
Because we are on a half space there is no external length scale, but $\sqrt{Dt}$ has units of length.

What about $x' = \frac{x}{\sqrt{Dt}}$?

Here we are not just scaling with parameters we are scaling independent variable t !

$\Rightarrow$ results in a new independent variable

$$\boxed{\eta = \frac{x}{\sqrt{4Dt}}}$$ Boltzmann variable



By introducing $\eta \sim \frac{x}{\sqrt{t}}$ we reduce PDE to an ODE !

η is called the similarity variable and the solution is said to be self-similar.

Q: What is the ODE?

First we scale $h' = \frac{h}{h_0}$ so that IC $h'=1$

Solution: $h'(x,t) = \Pi(\eta(x,t))$

Transform derivatives:

$$\frac{\partial h'}{\partial t} = \frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial t} \qquad \frac{\partial h'}{\partial x} = \frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial x}$$

$$\eta = \frac{x}{\sqrt{4Dt}} : \qquad \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4Dt}}, \qquad \frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}$$

$$\frac{\partial^2 h'}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{d\Pi}{d\eta}\frac{\partial \eta}{\partial x}\right) = \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{d^2\Pi}{d\eta^2} = \frac{1}{4Dt}\frac{d^2\Pi}{d\eta^2}$$

Substitute into PDE:

$$\frac{\partial h'}{\partial t} + D\frac{\partial^2 h'}{\partial x} = -\frac{\eta}{2\cancel{t}}\frac{d\Pi}{dt} - \frac{\cancel{D}}{4\cancel{D}\cancel{t}}\frac{d^2\Pi}{dt^2} = 0$$

$$\boxed{\begin{array}{lll} \text{ODE:} & \dfrac{d^2\Pi}{d\eta^2} + 2\eta \dfrac{d\Pi}{d\eta} = 0 & \eta \in [0,\infty) \\[2em] \text{BC:} & \Pi(0) = 0 & \lim\limits_{\eta\to\infty} \Pi = 1 \end{array}}$$

Solve ODE:

1) substitute: $u = \frac{d\Pi}{d\eta} \Rightarrow \frac{du}{d\eta} + 2\eta u = 0$

2) separate variables: $\frac{du}{u} = -2\eta\, d\eta$

$$\log u = -\eta^2 + a$$

$$u = c e^{-\eta^2}$$

3) resubstitute: $\frac{d\Pi}{d\eta} = c e^{-\eta^2}$

4) separate: $d\Pi = c e^{-\eta^2} d\eta$

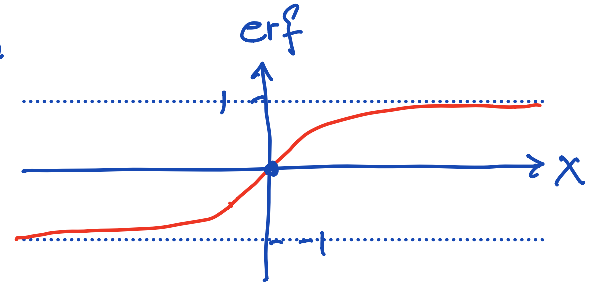
$$\Pi(\eta) = c\int_0^\eta e^{-z^2} dz \qquad z = \text{dummy var.}$$

does not have known analytic soln

$\Rightarrow$ give it a name and move on

5) Identify error function

$$\boxed{\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz}$$



Properties of error function:

- $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ "point symmetric"
- $\mathrm{erf}(0) = 0$
- $\mathrm{erf}(x) \approx x \qquad |x| \ll 1$
- $\lim_{x \to \infty} \mathrm{erf}(x) = 1$

Therefore: $\Pi(\eta) = c \frac{\sqrt{\pi}}{2} \mathrm{erf}(\eta)$

BC $\lim_{\eta \to \infty} \Pi(\eta) = c \frac{\sqrt{\pi}}{2} = 1 \qquad c = \frac{2}{\sqrt{\pi}}$

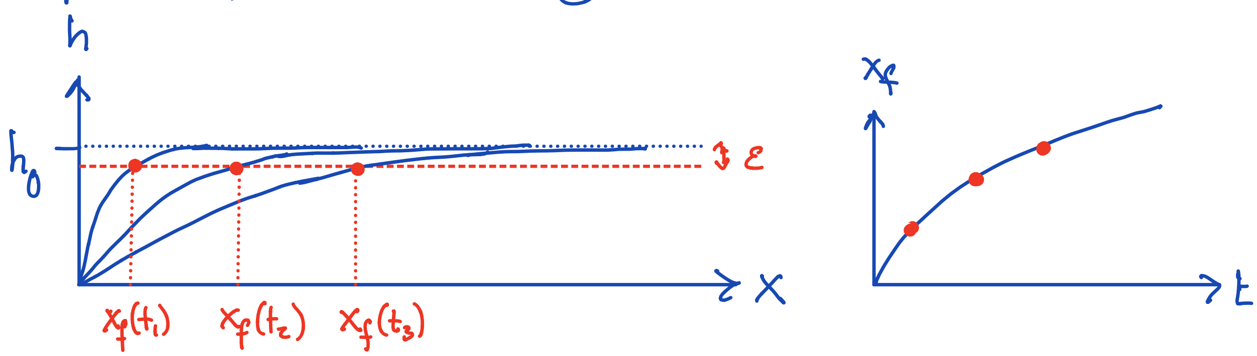
Self-similar solution: $\boxed{\Pi(\eta) = \mathrm{erf}(\eta)}$

6: Resubstitute: $h = h_0 h' \qquad \eta = \frac{x}{\sqrt{4Dt}}$

Transient evolution of head:

$$\boxed{h(x,t) = h_0 \,\mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right)}$$

Speed of front propagation



The front is defined as the location, x_f, where h has changed by εh_0 from its initial value h_0. We are looking for

$$h(x_f,t) = h_0 - \varepsilon h_0 = h_0(1-\varepsilon)$$

$$\Rightarrow \quad \cancel{h_0}(1-\varepsilon) = \cancel{h_0}\,\mathrm{erf}\left(\frac{x_f}{\sqrt{4Dt}}\right)$$

$$\frac{x_f}{\sqrt{4Dt}} = \mathrm{erf}^{-1}(1-\varepsilon) = \alpha(\varepsilon) = \text{const.}$$

$$\Rightarrow \quad \boxed{x_f = \alpha(\varepsilon)\sqrt{4Dt}}$$

Head perturbations propagate as $\underline{\sqrt{t}}$ in confined aquifers.