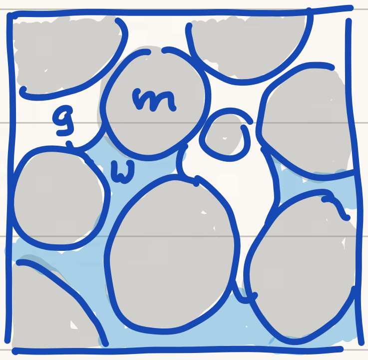
Variably Saturated Flow



Three phase system:

1) Solid matrix (m)

2) pore water (w)

3) pore gas (g)

Volume fractions:

$$\boxed{\theta_p = \frac{V_p}{V_T}}$$

V_p = volume of phase p

$V_T = V_w + V_s + V_g$ = total volume

$\Rightarrow \quad 0 \leq \theta_p \leq 1 \quad$ and $\quad \sum_p \theta_p = 1$

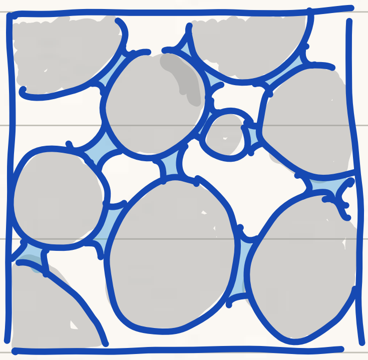
Porosity: $\boxed{\phi = \theta_w + \theta_g = 1 - \theta_m}$

Water content

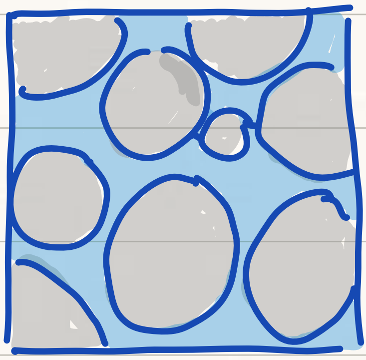
Key new variable in unsaturated flow is the (volumetric) water content $\theta_w \equiv \theta$

Varies in space and evolves in time: $\theta(\underline{x}, t)$

$\Rightarrow$ new dependent variable



θ_r is residual water content when water drains from a soil it leaves behind capillary bridges that cannot flow. (but they can evaporate)



$\theta_s = \phi$ saturated water content

$$\Rightarrow \boxed{\theta_r \leq \theta \leq \theta_s}$$

Rescale θ_w to vary from 0 to 1

$$\boxed{s = \frac{\theta - \theta_r}{\theta_s - \theta_r}}$$

water saturation
(effective water content)

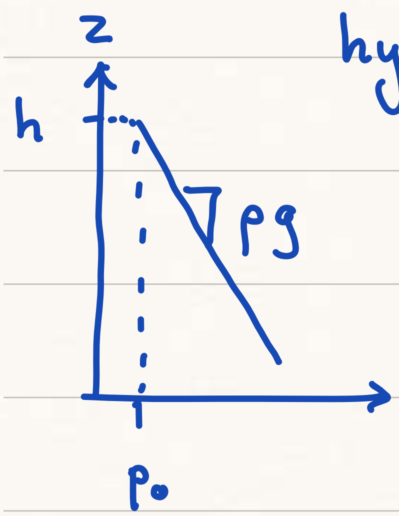
Darcy's law for unsaturated flow

Darcy's law in head form

$$\boxed{\underline{q} = -K\,\nabla h}$$

The head is a potential in the sense that water flows from high h to low h.

Relate h to pressure in saturated conditions



hydrostatic pressure:

$$p(z) = p_0 + \rho g\,(h - z)$$

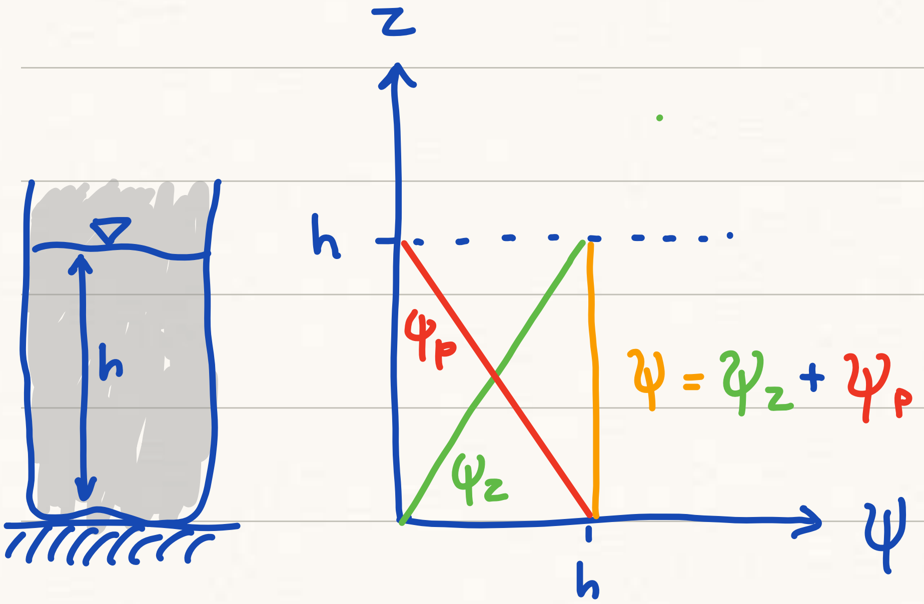
$$\Rightarrow \boxed{h = \frac{p - p_0}{\rho g} + z = \psi_p - \psi_z = \psi}$$

$h = \psi$ head or total potential

$\psi_z = z$ elevation head or gravitational potential

$\psi_p = \dfrac{p - p_0}{\rho g}$ pressure head or pressure potential

Variation of ψ's at equilibrium



$$\psi_p = \frac{p - p_0}{\rho g} = \frac{1}{\rho g}\left(p_0 + \rho g (h - z) - p_0\right) = h - z$$

This is for the case of a fully saturated medium !

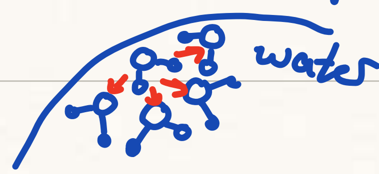
In unsaturated medium we have an additional potential !

Capillarity & Matric Potential

In unsaturated region gas is third phase.
Need new concepts:

Surface tension:

Near interface the hydrogen bonds generate a net attraction into the liquid.



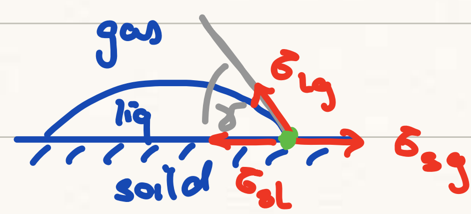
⇒ Water surface has a tendency to contract

⇒ Energy stored in form of surface tension

Energy per unit area: $\sigma_{wa} = 72.7 \ mJ/m^2$ (water-air)

Contact angle

Liquid in contact with solid in presence of gas.



contact angle is angle between sol-liq and liq-gas interfaces

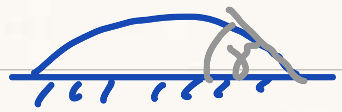
At the contact line the three surface tensions are in equilibrium.

$$\sigma_{sg} = \sigma_{sL} + \cos\gamma \, \sigma_{Lg}$$

rearranging

$$\boxed{\cos\gamma = \frac{\sigma_{sg} - \sigma_{sL}}{\sigma_{Lg}}}$$

where γ is the (eqbm) contact angle



$\gamma < 90°$

wetting fluid

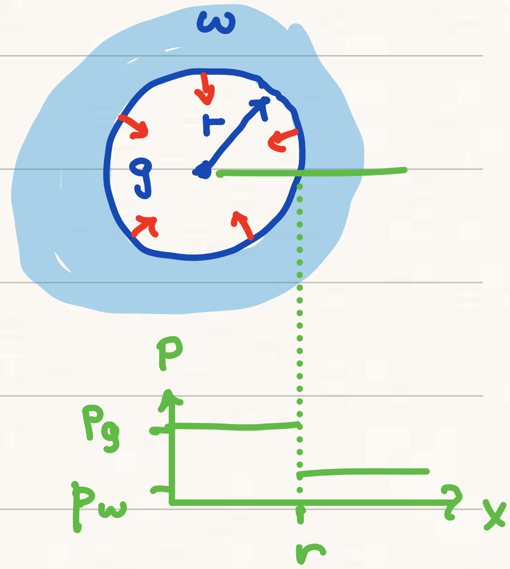
$\gamma > 90°$

non-wetting fluid

on most mineral surfaces water is a wetting fluid.

Capillary pressure

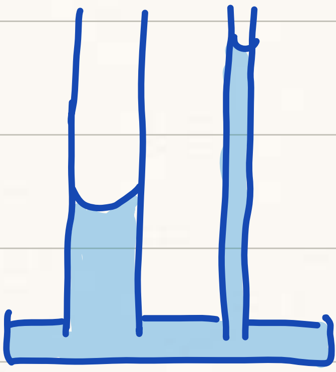
Surface tension creates a normal force on the phase on the concave side.

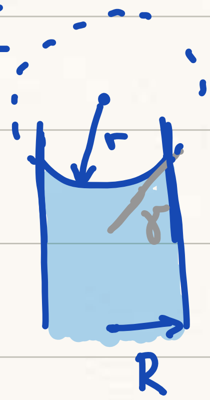


$$\boxed{p_g - p_w = p_c = \frac{2\sigma}{r}}$$

Young-Laplace Equation

Capillary rise

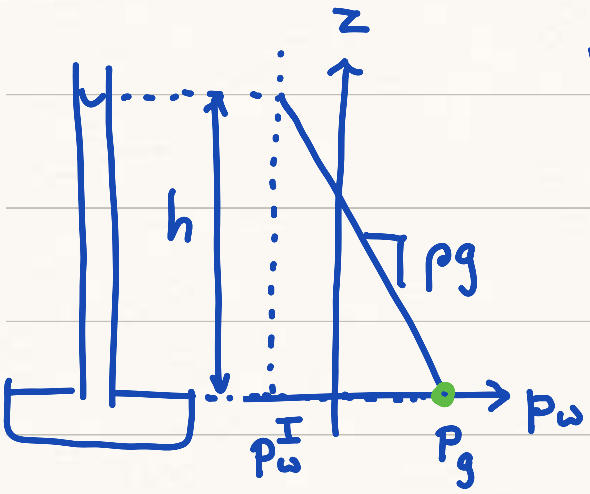




Given radius of tube, R and contact angle, γ, the radius of gas-water interface is:

$$r = \frac{R}{\cos\gamma}$$

Pressure of water at interface: $p_w^I = p_g - \frac{2\sigma\cos\gamma}{R}$



hydrostatic pressure in tube

$$p_w^I + \rho g h = p_g$$

subsitute

$$\cancel{p_g} - \frac{2\sigma\cos\gamma}{R} + \rho g h = \cancel{p_g}$$

$$\Rightarrow \quad \boxed{h = \frac{2\sigma\cos\gamma}{\rho g R}}$$ height of capillary rise

For water-air system:

$\gamma \approx 0°$ $\quad g = 9.81\,\frac{m}{s^2}$ $\quad \rho = 1000\,\frac{kg}{m^3}$ $\quad \sigma = 72.7\ mN/m$

$\Rightarrow \quad h = \frac{14.84}{R}$ $\quad$ R is in μm & h is in m

$\Rightarrow$ height of capillary rise depends inversely on radius

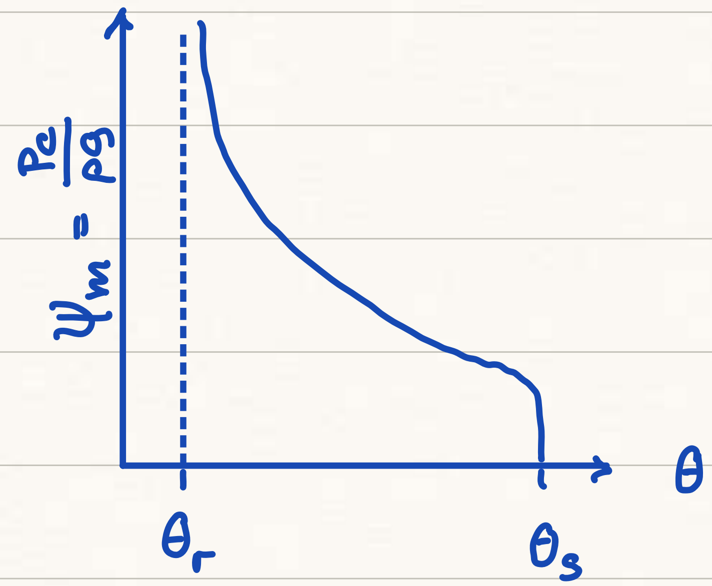
Height of capillary rise $\Rightarrow$ matric potential ψ_m

$$\boxed{h = \psi_m = \frac{p_c}{\rho g}}$$

Soil Water Characteristic (SWC)

Relationship between ψ_m and θ at eqbm.

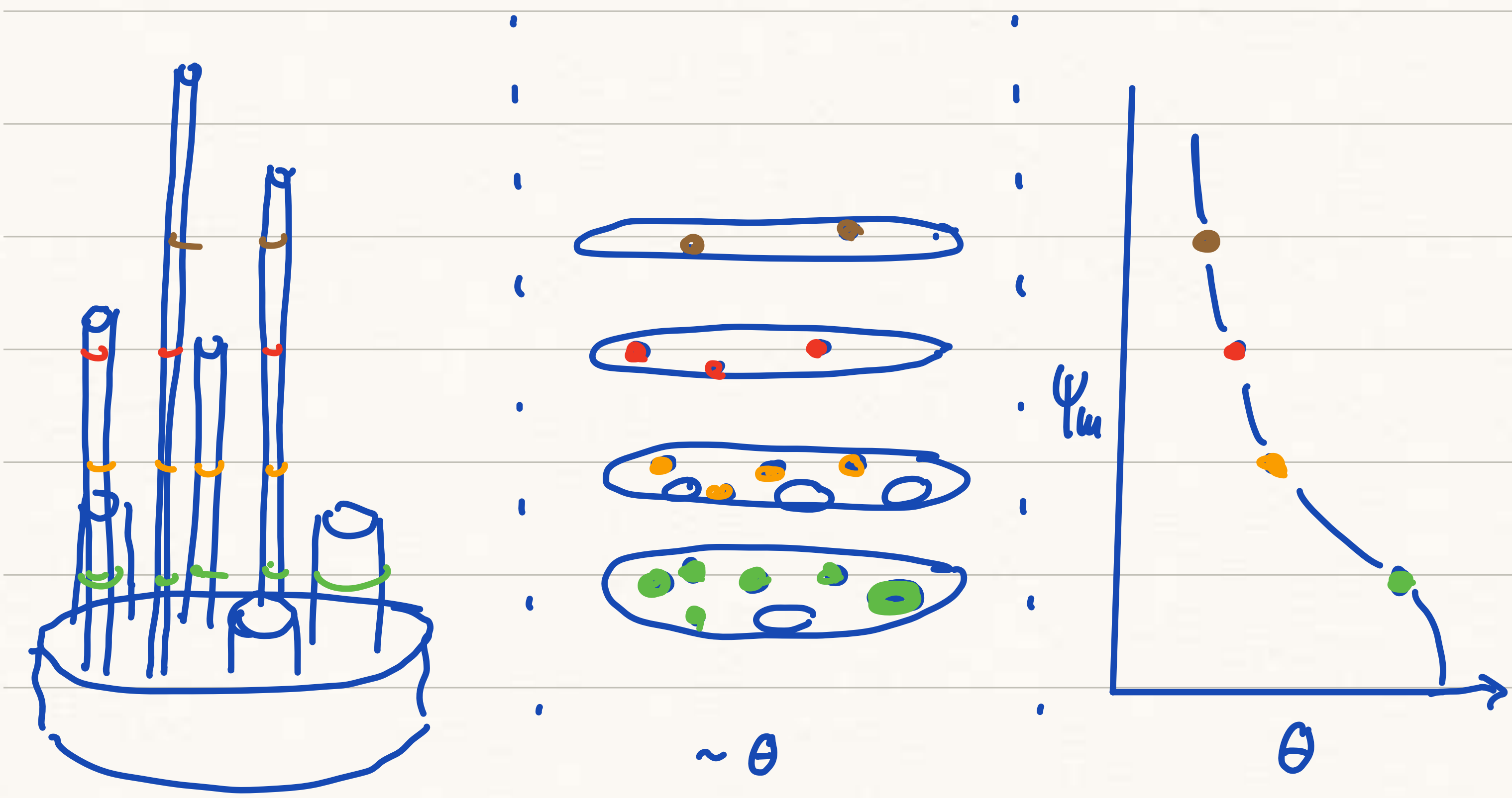
(Capillary pressure curve)



Primary hydraulic property for unsaturated flow.

Bundel of Capillary Tubes

Use ful conceptual model



Parametric models for SWC

Some classical models

1) Van Genuchten (1980) - VG

$$\boxed{s = \frac{1}{(1+(\alpha h)^n)^m}}$$ where $s = \frac{\theta - \theta_r}{\theta_s - \theta_r}$

model parameters: $\alpha, n, m, \theta_r, \theta_s$

θ_s can be measured

often assumed: $m = 1 - \frac{1}{n} = \frac{n-1}{n}$

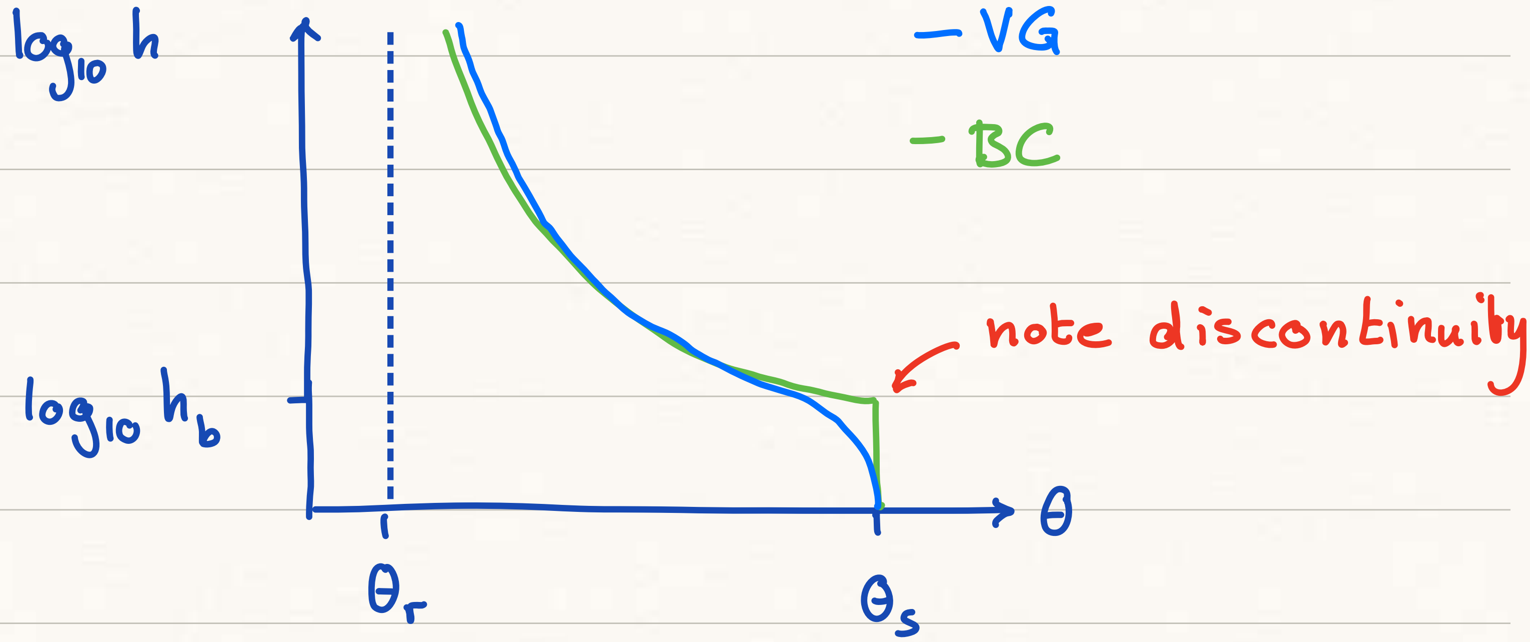
$$\Rightarrow \quad s = \frac{1}{1+(\alpha h)^{n-1}}$$ 3 param: α, n, θ_r

2) Brooks and Corey (1964) - BC

$$s = \left(\frac{h_b}{h}\right)^{\lambda} \qquad h > h_b$$

$$s = 1 \qquad h \le h_b$$

h is "bubbling pressure" / entry pressure



Typical values for silty loam:

VG: $\alpha = 0.417 \frac{1}{m}$ $\quad n = 1.75 \quad \theta_s = 0.513 \quad \theta_r = 0.05$

BC: $\lambda = 0.54 \quad \psi_b = 1.48\ m \quad \theta_s = 0.513 \quad \theta_r = 0.03$

Unsaturated Hydraulic Conductivity

Buckingham (1907)

K_s = saturated hyd. conductivity

$K_r(s)$ = relative hyd. conductivity

1) Mualem (1976) - Van Genuchten

$$\frac{K_r(s)}{K_s} = \sqrt{s}\left[1-\left(1-s^{1/m}\right)^m\right]^2$$

This can also be expressed in terms of matic head by substituting VG-SWC for s to obtain:

$$\frac{K_r(\psi_m)}{K_s} = \frac{\left[\left(1-(\alpha|\psi_m|)^{n-1}\left[1+(\alpha|\psi_m|)^n\right]^{-m}\right]^2}{\left[1+(\alpha|\psi_m|)^n\right]^{\frac{m}{2}}}$$

2) Brooks - Corey

$$\frac{K(s)}{K_s} = s^{3+\frac{2}{\lambda}} \qquad \frac{K(\psi_m)}{K_s} = \left(\frac{|\psi_m|}{\psi_b}\right)^{-2-3\lambda}$$

Darcy's law for unsaturated flow

$$q = -K(s)\nabla\psi = -K\nabla(h+z)$$

$$\boxed{q = -K(s)(\nabla h + 1)} \qquad h = |\psi_m|$$

Richards equation

Mass balance of water:

$$\frac{\partial}{\partial t}(\cancel{\rho}\theta) + \nabla\cdot(\cancel{\rho}q) = 0$$

$$\boxed{\frac{\partial\theta}{\partial t} - \nabla\cdot[K(s)(\nabla h+1)] = 0}$$

here we have θ, s, h

1) $K(s) \to K(\theta)$

$$\boxed{\frac{\partial\theta}{\partial t} - \nabla\cdot[K(\theta)(\nabla h+1)] = 0} \quad \text{mixed form}$$

Rewrite in terms of θ using

$$\nabla h(\theta) = \frac{dh}{d\theta} \nabla \theta$$

$$\frac{\partial \theta}{\partial t} - \nabla \cdot \left[K(\theta) \left(\frac{dh}{d\theta} \nabla \theta - 1 \right) \right]$$

introduce soil water diffusivity

$$D(\theta) = K(\theta) \frac{dh}{d\theta}$$

$$\boxed{\frac{\partial \theta}{\partial t} - \nabla \cdot \left[D(\theta) \left(\nabla \theta - K(\theta) \right) \right] = 0}$$ water content form

Or we can express $\theta = \theta(h)$

$$\frac{\partial \theta(h)}{\partial t} - \nabla \cdot \left[K(h) \left(\nabla h - 1 \right) \right]$$ mixed form

using chain rule

$$\boxed{c(h) \frac{\partial h}{\partial t} - \nabla \cdot \left[K(h) \left(\nabla h - 1 \right) \right] = 0}$$ head form

where $c(h) = \frac{d\theta}{dh}$