Dirichlet BC's & Constraints

Steady Geotherm

PDE : $-\nabla \cdot K \nabla T = \rho H_0 e^{-z/h_r}$ $\quad z \in [0, h]$

BC: $T(0) = T_s \quad T(h) = T_b$

$\Rightarrow$ heterogeneous (non-zero) Dirichlet BC's

Initially lets solve homogeneous problem

$$T(0) = T(h) = 0$$

not realistic but helpful step

Discretize PDE:

$$\underbrace{-\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}}_{\underline{\underline{L}}} * \underline{u} = \underline{fs}$$

$\underline{fs} = \rho H_0 e^{-\frac{xc}{h_r}}$

$\underline{\underline{Kd}} = \kappa \underline{\underline{I}}$ ← $Nfx \cdot Nfx$

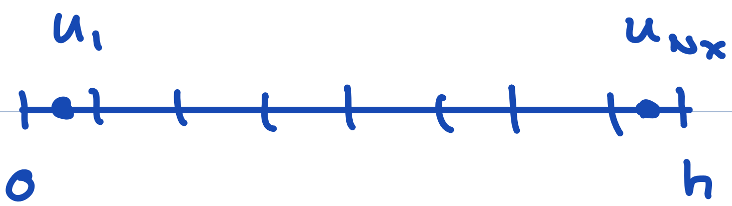
$$\Rightarrow \quad \boxed{\underline{\underline{L}}\, \underline{u} = \underline{fs}}$$

$\underline{\underline{L}} = -\underline{\underline{D}} * \underline{\underline{Kd}} * \underline{\underline{G}}$ "system matrix"

Matlab note: Use backslash to solve lin. sys!

$\underline{u} = \underline{\underline{L}} \backslash \underline{fs}$ (don't invert $\underline{\underline{L}}$)

Homogeneous BC's



$T(0) = T(h) = 0$

$\Rightarrow u_1 = 0 \;,\; u_{Nx} = 0$

note: the BC are imposed at cell centers

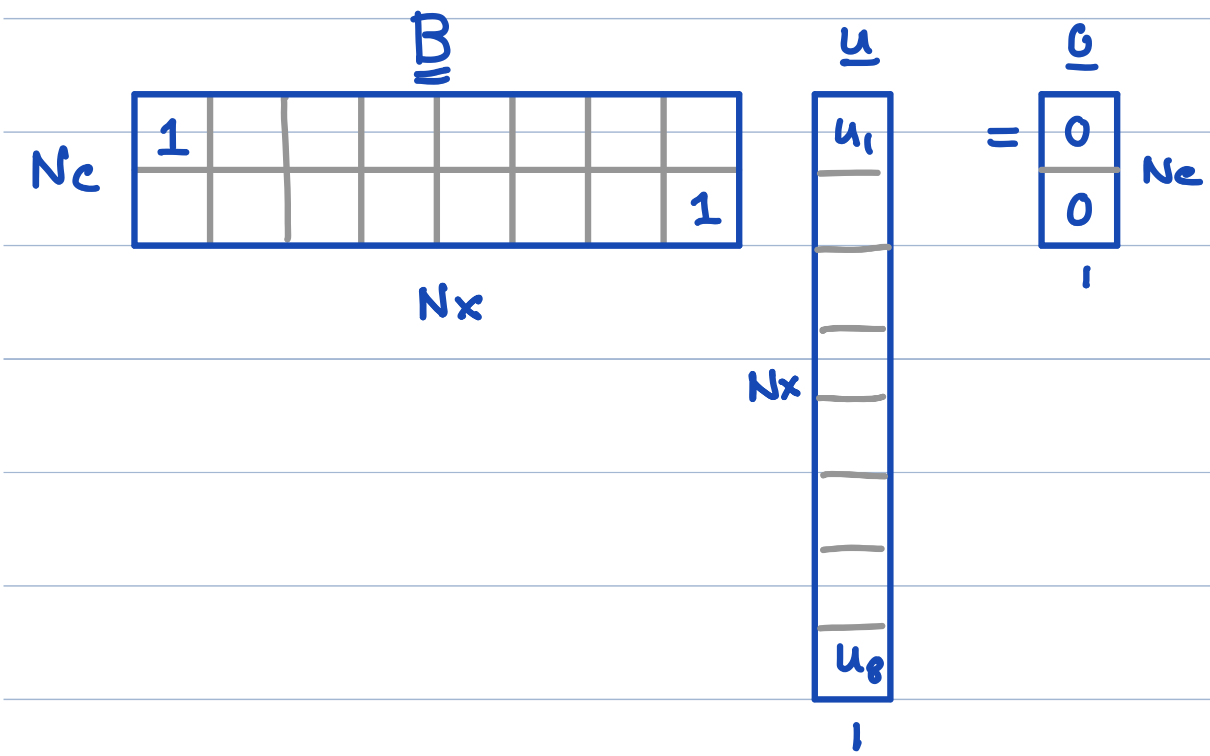
$\Rightarrow$ introduces an error

(in 1D we can just shift grid!)

BC's are Linear $\longrightarrow$ write as linear system

$\underline{\underline{B}} * \underline{u} = \underline{0}$ $\quad$ $\underline{\underline{B}}$ = "constraint matrix

$N_c \cdot N_x \quad N_x \cdot 1 \quad N_c \cdot 1$ $\quad$ N_c = number of constraints



Full discrete problem

PDE: $\underline{\underline{L}}\,\underline{u} = \underline{f_s}$ $\underline{\underline{L}} = N_x \cdot N_x$ system matrix

BC: $\underline{\underline{B}}\,\underline{u} = \underline{0}$ $\underline{\underline{B}} = N_c \cdot N_x$ constraint matrix

Neither system has a unique solution but together they do!

⇒ combine systems by eliminating constraints in $\underline{\underline{B}}$ from $\underline{\underline{L}}$.

Reduced Linear System

Constraints reduce number of unknowns to $N_x - N_c$

⇒ solve smaller system

reduced system: $\underline{\underline{L_r}}\,\underline{u_r} = \underline{f_{sr}}$

$\underline{u_r}$ is $N_x - N_c \cdot 1$ reduced solution vector

$\underline{f_{sr}}$ is $N_x - N_c \cdot 1$ reduced r.h.s. vector

$\underline{\underline{L_r}}$ is $(N_x - N_c) \cdot (N_x - N_c)$ reduced system matrix

What is relation between: $\underline{u_r} \longleftrightarrow \underline{u}$

$\underline{f_{sr}} \longleftrightarrow \underline{f_s}$

$\underline{\underline{L_r}} \longleftrightarrow \underline{\underline{L}}$

Projection matrix

Two vectors of different length are related by a rectangular matrix.

$$\underset{N_x \cdot 1}{\underline{u}} = \underset{N_x \cdot (N_x - N_c)}{\underline{\underline{N}}} * \underset{(N_x - N_c) \cdot 1}{\underline{u_r}}$$

What is $\underline{\underline{N}}$?

For now just require $\underline{\underline{N}}$ is orthonormal.

$$\underline{\underline{N}} = \begin{bmatrix} | & | & & | \\ \underline{n}_1 & \underline{n}_2 & \cdots & \underline{n}_{N_x - N_c} \\ | & | & & | \end{bmatrix}$$

$\underline{n}_i$ is the i-th column

$\underline{n}_i \cdot \underline{n}_i = 1$ (normal)

$\underline{n}_i \cdot \underline{n}_{j \neq i} = 0$ (ortho)

It follows:

a) $\underline{\underline{N}}^T \; \underline{\underline{N}} \; = \; \underline{\underline{I}}_r$

$(N_x - N_e) \cdot N_x \quad N_x \cdot (N_x - N_c) \quad (N_x - N_c) \cdot (N_x - N_c)$

b) $\underline{\underline{N}} \; \underline{\underline{N}}^T \; = \; \underline{\underline{I}}'$

$N_x \cdot (N_x - N_c) \quad (N_x - N_c) \cdot N_x \quad N_x \cdot N_x$

$\underline{\underline{I}}_r$ identity in reduced space

$\underline{\underline{I}}'$ "identity in full space but with Ne zeros on the diagonal

if $\underline{u} = \underline{\underline{N}} \; \underline{ur}$

$\underline{\underline{N}}^T \underline{u} = \underline{\underline{N}}^T \underline{\underline{N}} \; \underline{ur} = \underline{\underline{I}}_r \underline{ur} = \underline{ur}$

$\Rightarrow \underline{ur} = \underline{\underline{N}}^T \underline{u}$

$\underline{\underline{N}}$ allows transfer between full & reduced space

$$\boxed{\begin{aligned} \underline{u} &= \underline{\underline{N}} \; \underline{ur} \\ \underline{ur} &= \underline{\underline{N}}^T \underline{u} \end{aligned}}$$

we say that $\underline{\underline{N}}^T$ projects $\underline{u}$ into reduced space.

Similarly: $\underline{fs} = \underline{\underline{N}} \; \underline{fsr}$

$\underline{fsr} = \underline{\underline{N}}^T \underline{fs}$

How is $\underline{\underline{L}}$ projected into reduced space?

$$\underline{\underline{L}}\,\underline{u} = \underline{fs} \longrightarrow \underline{\underline{Lr}}\,\underline{ur} = \underline{fsr}$$

already know $\underline{u} = \underline{\underline{N}}\,\underline{ur}$ and $\underline{fs} = \underline{\underline{N}}\,\underline{fsr}$

$$\Rightarrow \underline{\underline{L}}\,\underline{\underline{N}}\,\underline{ur} = \underline{\underline{N}}\,\underline{fsr}$$

need to remove $\underline{\underline{N}}$ on l.h.s.

$$\underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}}\,\underline{ur} = \underbrace{\underline{\underline{N}}^T\underline{\underline{N}}}_{\underline{\underline{Ir}}}\,\underline{fsr} = \underline{fsr}$$

$$\underbrace{\underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}}}_{\underline{\underline{Lr}}}\,\underline{ur} = \underline{fsr}$$

Hence we have:

$\underline{\underline{Lr}}\,\underline{ur} = \underline{fsr}$ where

$$\underline{\underline{Lr}} = \underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}}$$
$$\underline{ur} = \underline{\underline{N}}^T\underline{u}$$
$$\underline{fsr} = \underline{\underline{N}}^T\underline{fs}$$

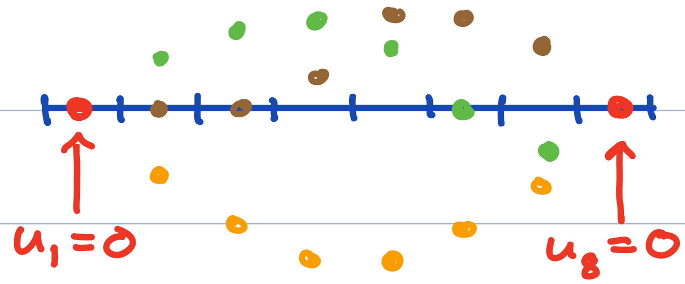
Just need to find $\underline{\underline{N}}$!

$\underline{\underline{N}}$ needs to contain information about BC's.

$\Rightarrow$ related to $\underline{\underline{B}}$

Constraint System: $\underline{\underline{B}}\,\underline{u} = \underline{0}$

$\Rightarrow$ many solutions



- soln 1
- soln 2
- soln 3

} Nullspace of $\underline{\underline{B}}$: $\mathcal{N}(\underline{\underline{B}})$

$\Rightarrow$ search for solutions in $\mathcal{N}(\underline{\underline{B}})$!

$\mathcal{N}(\underline{\underline{B}})$ is reduced space

Matrix $\underline{\underline{N}}$ is any orthonormal basis for $\mathcal{N}(\underline{\underline{B}})$

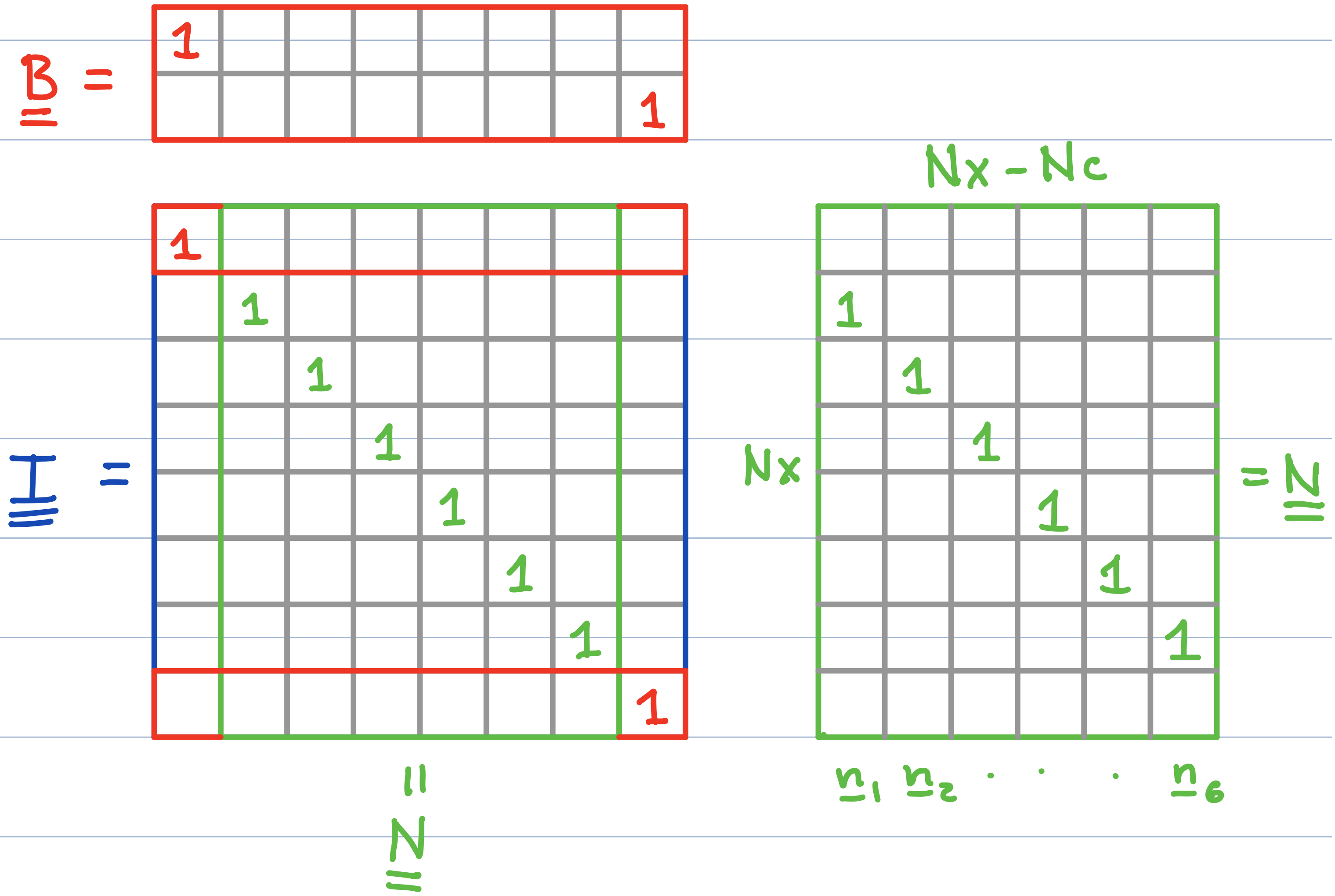
In Matlab: $\underline{\underline{N}}$ = null($\underline{\underline{B}}$)

$\underline{\underline{N}}$ = spnull($\underline{\underline{B}}$)

$\rightarrow$ download Matlab Central!

Problem: General but expensive.

Simple constraints $\rightarrow$ $\underline{\underline{N}}$ easy to find



⇒ spliting $\underline{\underline{I}}$ into $\underline{\underline{B}}$ & $\underline{\underline{N}}$

Why is $\underline{\underline{N}}$ basis for $\mathcal{N}(\underline{\underline{B}})$?

$$\underline{u} = \sum_{i=1}^{Nx-Nc} \alpha_i \underline{n}_i \in \mathcal{N}(\underline{\underline{B}})$$

$\alpha_i \in \mathbb{R}$ weight

$\underline{n}_i$ = i-th column of $\underline{\underline{N}}$

because $u(1) = u(Nx) = 0$ for all α_i

In Matlab:

1) Vector containing Dirichlet cells:

```
BC.dof_dir = [Grid.dof_xmin; Grid.dof_xmax];
```

2) Build $\underline{\underline{B}}$ selecting rows from $\underline{\underline{I}}$

```
B = I(BC.dof_dir, :);
```

3) Build $\underline{\underline{N}}$ deleating columns from $\underline{\underline{I}}$

```
N = I;
N(:, BC.dof_dir) = [];
```

Heterogeneous BC's

Return to original geotherm problem

PDE: $-\nabla \cdot K \nabla T = \rho H_0 e^{-z/h_r}$ $\quad z \in [0, h]$

BC: $T(0) = T_s \quad T(h) = T_b$

Discrete systems

$$\underline{\underline{L}}\,\underline{u} = f_s$$
$$\underline{\underline{B}}\,\underline{u} = g$$

$$g = \begin{bmatrix} T_s \\ T_b \end{bmatrix}$$

$\underline{\underline{B}}$ same as before → location of BC's

Because $\underline{\underline{B}}\,\underline{u} = g$ is linear

⇒ decompose: $\underline{u} = \underline{u}_0 + \underline{u}_p$

homogeneous soln.: $\underline{\underline{B}}\,\underline{u}_0 = \underline{0}$

particular soln.: $\underline{\underline{B}}\,\underline{u}_p = g$

$$\left.\begin{array}{l}\underline{\underline{B}}\,\underline{u}_0 = \underline{0}\\ \underline{\underline{B}}\,\underline{u}_p = g\end{array}\right\}\ \underline{\underline{B}}\,(\underbrace{\underline{u}_0 + \underline{u}_p}_{\underline{u}}) = g$$

Note: $\underline{u}$ is unique (assuming suitable BC's)

split into $\underline{u}_0 + \underline{u}_p$ is not unique

→ simplest (obvious) choice

Two questions: 1) How do we find $\underline{u}_p$?

2) Given $\underline{u}_p$ how do we find associated $\underline{u}_0$?

Start with Q2: Suppose we know $\underline{u}_p$

$\underline{\underline{L}}\,(\underline{u}_0 + \underline{u}_p) = \underline{f_s}$ $\qquad$ $\underline{u}_p$ is known → r.h.s

$\underline{\underline{L}}\,\underline{u}_0 = \underline{f_s} - \underbrace{\underline{\underline{L}}\,\underline{u}_p}$ $\qquad$ $\underline{\underline{L}}\,\underline{u}_0 = \underline{f_s} + \underline{f_d}$ $\qquad$ $\underline{f_d} = -\underline{\underline{L}}\,\underline{u}_p$

$\underline{f_d}$ = Dirichlet r.h.s.

Form reduced system as before:

$$\underline{\underline{L}}\,\underline{u}_{0r} = \underline{f}_r \qquad \underline{f}_r = \underline{\underline{N}}^T(\underline{f}_s + \underline{f}_d)$$

where $\underline{u}_{0r}$ is reduced homogeneous solution

$$\underline{u}_0 = \underline{\underline{N}}\,\underline{u}_{0r}$$

$$\Rightarrow \quad \underline{u} = \underline{u}_0 + \underline{u}_p \qquad \text{full solution}$$

Q1: How do we find $\underline{u}_p$?

$\underline{u}_p$ is not unique: $\underline{\underline{B}}\,\underline{u}_p = \underline{g}$

$\underline{\underline{L}}\,\underline{u}_p \neq \underline{f}_s$ does no satisfy PDE by itself

$\Rightarrow$ just satisfy $\underline{\underline{B}}\,\underline{u}_p = \underline{g}$!

Simplest solution: $\underline{u}_p = \begin{bmatrix} T_s \\ 0 \\ 0 \\ \vdots \\ 0 \\ T_b \end{bmatrix}$

$\underline{\underline{B}} \rightarrow$ Location & $\underline{g} \rightarrow$ value

Matlab: `up(dof_dir) = g`

Simple because our constraints are simple.

General approach for arbitrary constraints

$\Rightarrow$ solve reduced system for $\underline{u_p}$

should be a $N_c \cdot N_c$ system!

$$\underset{N_c \cdot 1}{\underline{u_{pr}}} = \underset{N_c \cdot N_x}{\underline{\underline{B}}}\ \underset{N_x \cdot 1}{\underline{u_p}} \quad \longrightarrow \quad \underset{N_x \cdot 1}{\underline{u_p}} = \underset{N_x \cdot N_c}{\underline{\underline{B}}^T}\ \underset{N_c \cdot 1}{\underline{u_{pr}}}$$

substitute:

$$\underline{\underline{B}}\ \underline{u_p} = \underline{g}$$

$$\underset{N_c \cdot N_x}{\underline{\underline{B}}}\ \underset{N_x \cdot N_c}{\underline{\underline{B}}^T}\ \underline{u_{pr}} = \underline{g} \qquad\qquad \underset{N_c \cdot N_c}{\underline{\underline{B}}\,\underline{\underline{B}}^T}$$

Particular reduced system: $\boxed{\underline{\underline{B}}\,\underline{\underline{B}}^T\,\underline{u_r} = \underline{g}}$

Summary of Dirchlet BC implementation

Two new functions:

1) [B, N, fn] = build_bnd(BC, Grid, I)

2) u = solve_lbvp(L, f, B, g, N)

$\Rightarrow$ next home work

General implementation

works for 1D, 2D, 3D, cylindrical, spherical, ...

Inside solve_lbvp.m

1) Find $\underline{u}_p$

$$\underline{\underline{B}}\,\underline{\underline{B}}^T \underline{u}_{pr} = \underline{g} \longrightarrow \underline{u}_p = \underline{\underline{B}}^T \underline{u}_{pr}$$

2) Find associated homogeneous solution

$$\underline{\underline{L}}_r \underline{u}_{0r} = \underline{f}_r \longrightarrow \underline{u}_0 = \underline{\underline{N}}\,\underline{u}_{0r}$$

3) Full solution: $\underline{u} = \underline{u}_0 + \underline{u}_p$