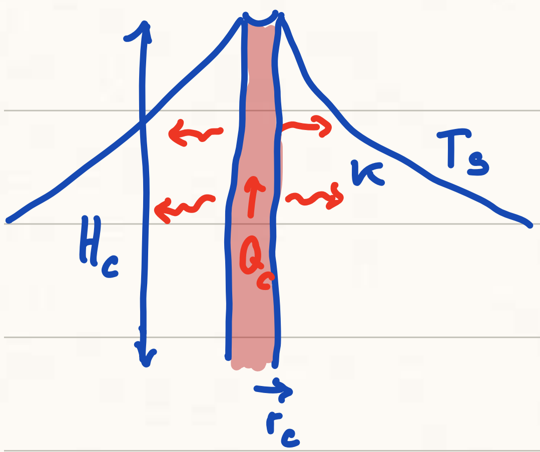
Conservative Finite Differences

Aim: Many finite difference approaches
$\Rightarrow$ motivate our choice

Example problem:

Steady heat flow from volcanic conduit



T_s = surface temp.

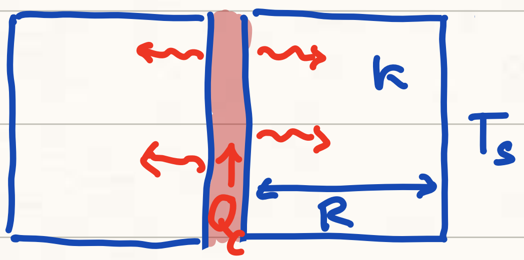
Q_c = heat flow from conduit

r_c = conduit radius

R = radius of volcano

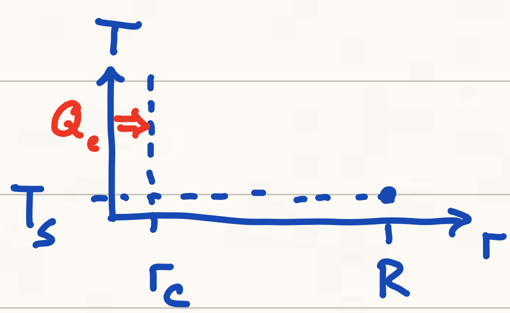
k = thermal conductivity

Complicated 2D geometry $\Rightarrow$ simplified 1D model



"Spherical cow" volcano

(we can make geometry more complicated later)



1D model

Heat equation

$$\cancel{\rho c_p \frac{\partial T}{\partial t}}^{\,0} - \nabla \cdot [\kappa \nabla T] = 0$$ steady state

$$- \nabla \cdot [\cancel{\kappa}^{\,0} \nabla T] = 0$$ $\kappa =$ constant

$$\nabla \cdot \nabla T = 0$$

$$\underbrace{\frac{1}{r} \frac{d}{dr}}_{\nabla \cdot} \left(r \underbrace{\frac{dT}{dr}}_{\nabla} \right) = 0$$ radial coord.

PDE: $\frac{d}{dr}\left(r \frac{dT}{dr} \right) = 0 \qquad r \in [r_c, R]$

Boundary conditions:

1) $T(R) = T_s$ fix T at surface (Dirichlet BC)

2) $Q_c = A_c q \qquad A_c = 2\pi r_c H_c$

$Q_c = - A_c \kappa \frac{dT}{dr} \qquad q = -\kappa \nabla T = -\kappa \frac{dT}{dr}$

$$\Rightarrow \frac{dT}{dr} = - \frac{Q_c}{A_c \kappa}$$

Summary

PDE: $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \qquad r \in [r_c, R]$

BCs: $T(R) = T_s$

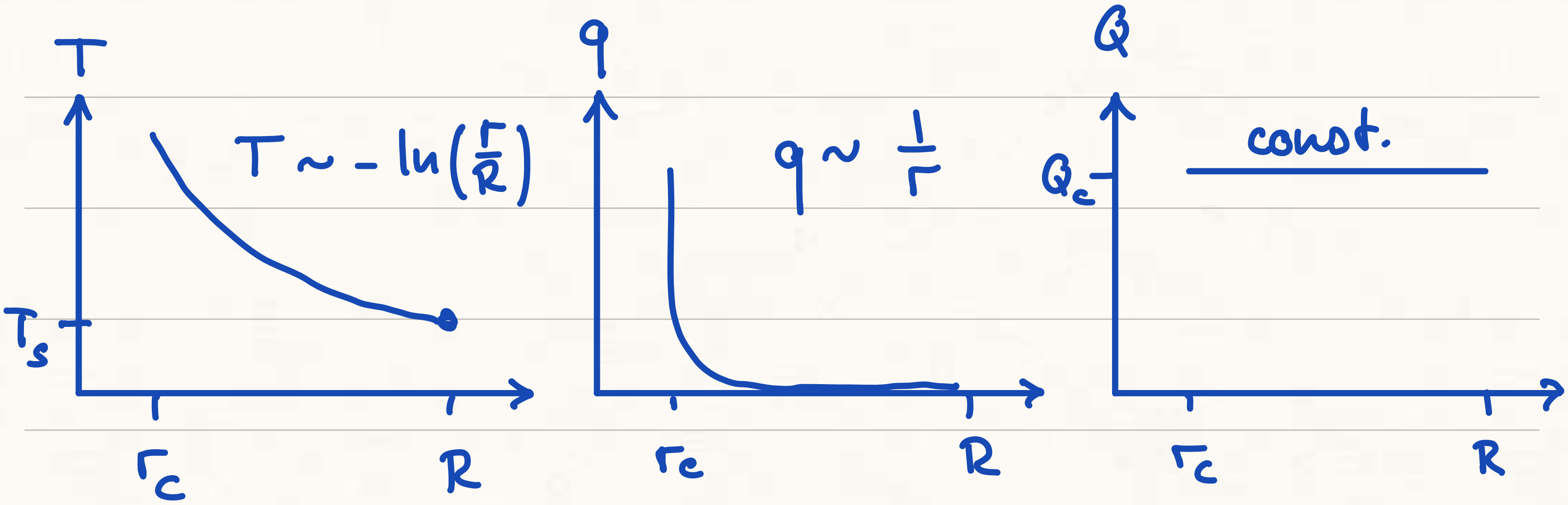
$\left.\frac{dT}{dr}\right|_{r_c} = -\frac{Q_c}{A_c k}$

Analytic solution:

$$T(r) = T_s - \frac{Q_c}{2\pi H_c k}\ln\left(\frac{r}{R}\right)$$

$$q(r) = \frac{Q_c}{2\pi H}\frac{1}{r}$$

$$Q(r) = Q_c$$



If $r_c \ll R$ difficult problem $\left.\frac{dT}{dr}\right|_{r_c} \to \infty$

$\Rightarrow$ boundary layer at r_c

Finite Difference Solution

Here we focus on discretization of PDE and gloss over the BC's!

Attempt 1:

PDE: $\frac{d}{dr}\left(r \frac{dT}{dr}\right) = r \frac{d^2T}{dr^2} + \frac{dT}{dr} = 0$

Use standard FD approximations

$$\frac{dT}{dr} = \frac{T_{i+1} - T_{i-1}}{2\Delta r} = \underline{\underline{D}}\,\underline{T}$$

$$\frac{d^2T}{dr^2} = \frac{T_{i+1} - 2T_i + T_{i+1}}{\Delta r^2} = \underline{\underline{D}}^2\,\underline{T}$$

$\underline{\underline{D}}$ = differentiation matrix

Observations:

- leads to large errors that dissapear
- slowly as we refine grid!
  due to the boundary layer
- energy is <u>not</u> conserved! ($Q \neq$ constant)

$\Rightarrow$ need improve conservation!

Attempt 2:

Previous attempt :

- chain rule "destroyed" divergence
- divergence comes from balance law

$\Rightarrow$ discretize with divergence in tact ∇!

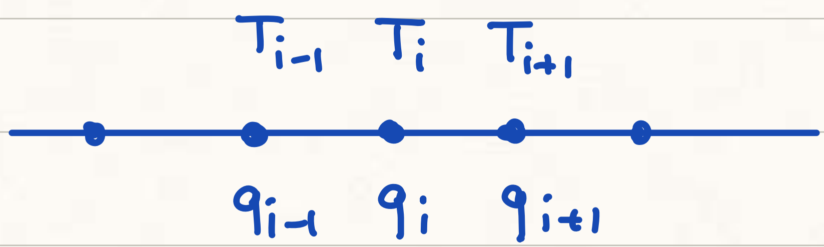
PDE: $-\nabla \cdot [\kappa \nabla T] = f_s$

Write as "div-grad" system of equations

1) $\nabla \cdot q = f_s$ mass balance

2) $q = -\kappa \nabla T$ Fourier's law

$\Rightarrow$ introduce q as new variable



co-located

In 1D linear geometry

1) $\nabla \cdot q = f_s \xrightarrow{1D} \dfrac{dq}{dx} = f_s \xrightarrow{FD} \dfrac{q_{i+1} - q_{i-1}}{2\Delta x} = f_i$

2) $q = -\kappa \nabla T \xrightarrow{1D} q = -\kappa \dfrac{dT}{dx} \xrightarrow{FD} q_i = -\kappa \dfrac{T_{i+1} - T_{i-1}}{2\Delta x}$

Either solve for q_i & T_i together

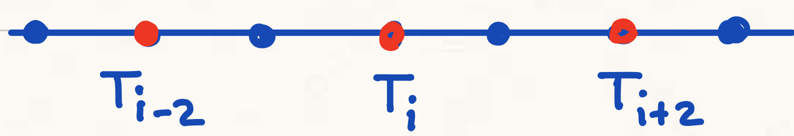
$\Rightarrow$ more expensive

or eliminate q_i by substituting ② into ①

$$① \quad \frac{q_{i+1} - q_{i-1}}{2\Delta x} = f_i$$

$$\frac{1}{2\Delta x}\Big[\underbrace{-\kappa\frac{T_{i+2} - T_i}{2\Delta X}}_{q_{i+1}} - \underbrace{\left(-\kappa\frac{T_i + T_{i-2}}{2\Delta x}\right)}_{q_{i-1}}\Big] = f_i$$

simplify: $$\underbrace{\frac{1}{4\Delta x^2}\left(-T_{i+2} + 2T_i - T_{i-2}\right)}_{\text{stencil of FD approximation}} = f_i$$



leads to a wide stencil

$\Rightarrow$ decoupling of even & odd nodes

$\Rightarrow$ oscillations

But despite oscillations error decreases!

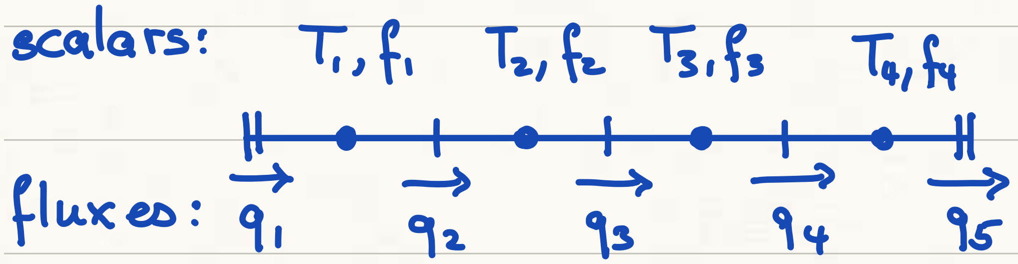
Attempt 3: Conservative Finite Differences

Aims: 1) Conserve mass/energy

2) Compact stencil

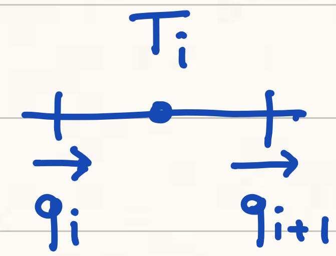
3) Avoid decoupling

$\Rightarrow$ staggered grid



$\Rightarrow$ T_i and q_i are <u>not</u> co-located

Control Volume / Cell :



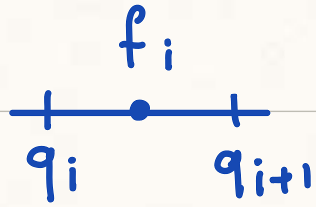
$$\left.\begin{array}{l} \text{Energy in cell: } \rho c_p T_i \\ \text{Fluxes in/out of cell: } q_i \ \& \ q_{i+1} \end{array}\right\} \text{conservation}$$

Div-Grad discretization:

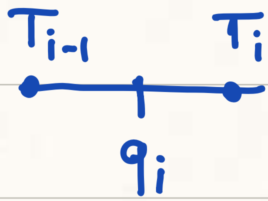
1) $\nabla \cdot q = f_s \xrightarrow{1D} \frac{dq}{dx} = f_s \xrightarrow{CFD} \frac{q_{i+1} - q_i}{\Delta x} = f_i$

2) $q = -\kappa \nabla T \xrightarrow{1D} q = -\kappa \frac{dT}{dx} \xrightarrow{CFD} q_i = -\kappa \frac{T_i - T_{i-1}}{\Delta x}$

Note:



$$\frac{q_{i+1}-q_i}{\Delta x} = f_i$$



$$q_i = -k\,\frac{T_i - T_{i-1}}{\Delta x}$$

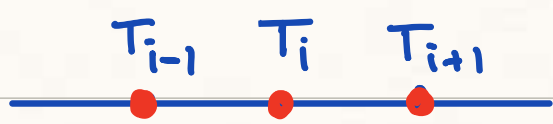
$\Rightarrow$ central differences due to staggering

Eliminate q_i by substituting ② into ①

① $$\frac{q_{i+1}-q_i}{\Delta x} = f_i$$

$$\frac{1}{\Delta x}\left[-k\,\frac{T_{i+1}-T_i}{\Delta x} - \left(-k\,\frac{T_i - T_{i-1}}{\Delta x}\right)\right] = f_i$$

$$\frac{k}{\Delta x^2}\underbrace{\left(-T_{i+1} + 2T_i - T_{i-1}\right)}_{\text{CFD stencil}} = f_i$$



compact stencil

$\Rightarrow$ gives exact solution for example problem

$\Rightarrow$ generally much better solution