Lecture 12: Transient heat Conduction

Logistic: HW4 due

HW5 → due Th (radial coord.)

Last time: - Analytic Jacobian

- Radial coordinates

Today: Transient heat conduction

⇒ time evolution

Energy balance:

PDE: $\underbrace{\rho c_p \frac{\partial T}{\partial t}}_{\text{new term}} - \nabla \cdot [k \nabla T] = f_s$

BC: $T(H) = T_s$ $\quad q \cdot n|_{z=0} = -|q_m|$

IC: $T(z, t=0) = T_0(z)$ initial condition (new)

Discretize PDE:

PDE: $\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot \kappa \nabla T = f_s$

? $\quad \underbrace{- \underline{\underline{D}}\ \underline{\underline{Kd}}\ \underline{\underline{G}}}_{\underline{\underline{L}} = \text{linear transport operator}}\ \underline{u} = \underline{f_s}$

New term (accumulation term):

1) time derivative: $\frac{\partial T}{\partial t} \approx \frac{\underline{u}^{n+1} - \underline{u}^{n}}{\Delta t}$

2) coefficients: $\rho(\underline{x})\, c_p(\underline{x}) = \underset{Nx \cdot 1}{\underline{rho}} .* \underset{Nx \cdot 1}{\underline{cp}}$

$\Rightarrow \quad \underline{\underline{S}} = \text{spdiags}(\underline{rho .* cp}, 0, Nx, Nx)$

substitute:

$$\underline{\underline{S}}\,\underline{(\underline{u}^{n+1} - \underline{u}^{n})} + \Delta t\, \underline{\underline{L}}\, \underline{u}^{?} = \Delta t\, \underline{f_s} + \Delta t\, \underline{f_n}$$

$\underline{u}^{\theta}$

Theta Method

$\underline{u}^{\theta} = \theta\, \underline{u}^{n} + (1-\theta)\, \underline{u}^{n+1}$ $\qquad \theta$ = parameter

substitute

$$\underline{\underline{S}}(\underline{u}^{n+1} - \underline{u}^{n}) + \Delta t\, \underline{\underline{L}}[\theta \underline{u}^{n} + (1-\theta)\underline{u}^{n+1}] = \Delta t\, \underline{f_s}$$

$\underline{u}^{n}$ is known $\longrightarrow$ r.h.s

$\underline{u}^{n+1}$ is unknown $\longrightarrow$ l.h.s.

$$\underbrace{[\underline{\underline{S}} + \Delta t(1-\theta)\underline{\underline{L}}]}_{\underline{\underline{IM}}}\, \underline{u}^{n+1} = \underbrace{[\underline{\underline{S}} - \Delta t\theta \underline{\underline{L}}]}_{\underline{\underline{EX}}}\, \underline{u}^{n} + \Delta t\, \underline{f_s}$$

Linear system for n-th time step:

$$\boxed{\underline{\underline{IM}}\, \underline{u}^{n+1} = \underline{\underline{EX}}\, \underline{u}^{n} + \Delta t\, \underline{f_s}}$$

Implicit matrix: $\underline{\underline{IM}} = \underline{\underline{S}} + \Delta t(1-\theta)\underline{\underline{L}}$

explicit matrix: $\underline{\underline{EX}} = \underline{\underline{S}} - \Delta t\, \theta \underline{\underline{L}}$

$$(\underline{\underline{L}} = -\underline{\underline{D}}\, \underline{\underline{K_d}}\, \underline{\underline{G}})$$

Choice of θ $\rightarrow$ determine method of time integration

Properties of θ-Method

$\theta = 1$: Forward Euler Method

$\underline{\underline{M}} = \underline{\underline{S}} \Rightarrow$ diagonal $\Rightarrow$ cheap to invert

$$\underline{u}^{n+1} = \underline{\underline{S}}^{-1}(\Delta t\, \underline{f_s} + \underline{\underline{EX}}\, \underline{u}^{n})$$

$\Rightarrow$ explicit method to determine u^{n+1}

↳ (no lin. sys. to solve)

→ first-order accurate

$$(\underline{u}^{n+1} - \underline{u}^{n}) + \Delta t\, \underline{\underline{L}}\, \underline{u}^{n} = \Delta t\, f_s$$

$\Rightarrow$ one sided approx. of time deriv.

→ conditionally stable: $\boxed{\Delta t \leq \frac{\Delta x^2}{2\alpha}}$

$\alpha = \frac{k}{\rho c_p}$ thermal diffusivity

$\Rightarrow$ many timesteps

$\theta = 0$: Backward Euler Method

$\underline{\underline{M}}$ not diagonal $\Rightarrow$ solve lin. sys at every timestep

$\Rightarrow$ unconditionally stable

→ first-order (onesided approx.)

$\Theta = \frac{1}{2}$: Crank-Nicholson Method

$\underline{\underline{IM}}$ not diagonal $\Rightarrow$ solve lin. sys.

-> second order in time because it is centered in time.

-> unconditionally stable (oscillation limit)

Amplification Matrix

$\underline{f_S} = \underline{0}$

at n-th time step: $\underline{\underline{IM}}\,\underline{u}^{n+1} = \underline{\underline{EX}}\,\underline{u}^{n}$

$$\underline{u}^{n+1} = \underbrace{\underline{\underline{IM}}^{-1}\underline{\underline{EX}}}_{\underline{\underline{A}}}\,\underline{u}^{n}$$

$\underline{\underline{A}}$ = amplification matrix

$$\underline{u}^{n+1} = \underline{\underline{A}}\,\underline{u}^{n}$$

$$\underline{u}^{n} = \underline{\underline{A}}\,\underline{u}^{n-1}$$

$$\underline{u}^{n+1} = \underline{\underline{A}}\,u^{n-2}$$

$\Rightarrow \boxed{\underline{u}^{n} = \underline{\underline{A}}^{n}\,\underline{u}^{0}}$ $\quad \underline{u}^{0}$ initial condition n-th power of a matrix

spectral decomposition: $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^{-1}$

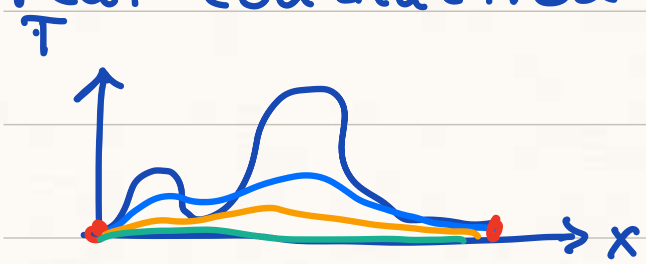
$\underline{\underline{Q}}$ = matrix of eigenvectors

$\underline{\underline{\Lambda}}$ = diagonal matrix of eigenvalues

$$\underline{\underline{A}}^2 = \underline{\underline{A}}\,\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underbrace{\underline{\underline{Q}}^{-1}\underline{\underline{Q}}}_{\underline{\underline{I}}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}^2\underline{\underline{Q}}^{-1}$$

$$\Rightarrow \quad \boxed{\underline{\underline{A}}^n = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}^n\underline{\underline{Q}}^{-1}}$$

Heat conduction problem:



from physical intuition we expect IC to decay to zero (in absence of sources)

$\Rightarrow$ for decay $|\lambda(\underline{\underline{A}})| \leq 1$

Look at $\lambda(\Delta t)$ to find time step limit!

Instead of numerical analysis
we compute $\underline{\underline{A}}$ and look at its eigenvalues!