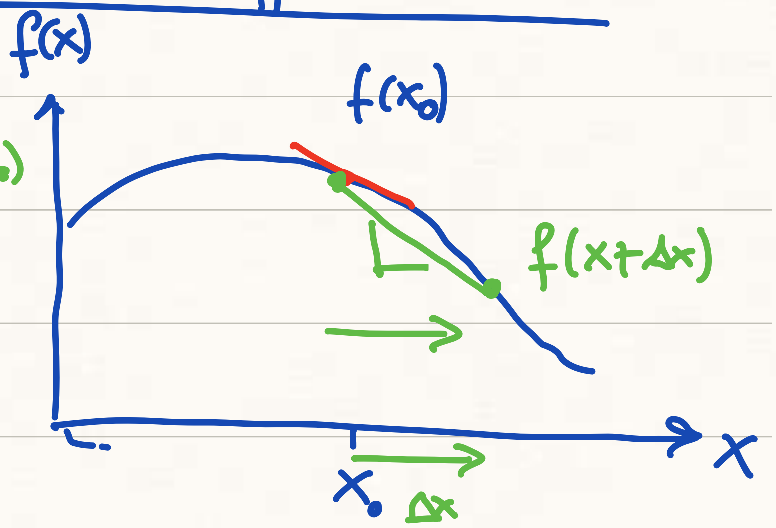
Lecture 2: Intro to finite differences

In calculus:

$$\dot{f}(x_0) = \left.\frac{df}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$



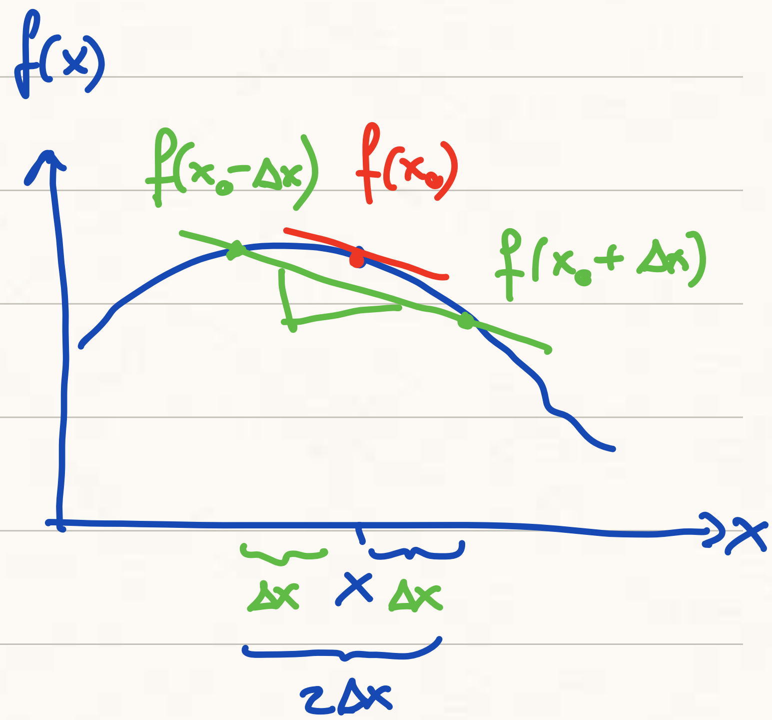
Finite difference approx.:

$$\hat{\dot{f}}(x_0) = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + O(\Delta x)$$

⇒ first-order approx.

⇒ one-sided approx.

Central difference approx:



$$\boxed{\hat{\dot{f}} = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\,\Delta x}}$$

⇒ second-order approx.

good balance between accuracy & complexity/cost

typically we take second derivatives

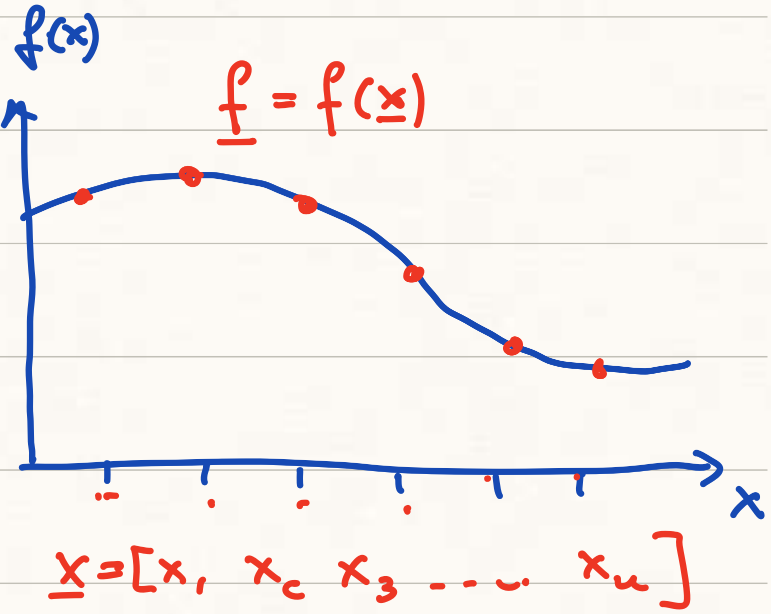
Differentiation matrix

Discrete equivalent of a function :→ vectors

$$\underline{f} = f(\underline{x})$$

Similarly for derivative

$$\underline{df} = \dot{f}(\underline{x})$$



Derivative is a linear differential operator

$$\dot{f}(x) = \mathcal{D}(f(x))$$

↑ derivative operator

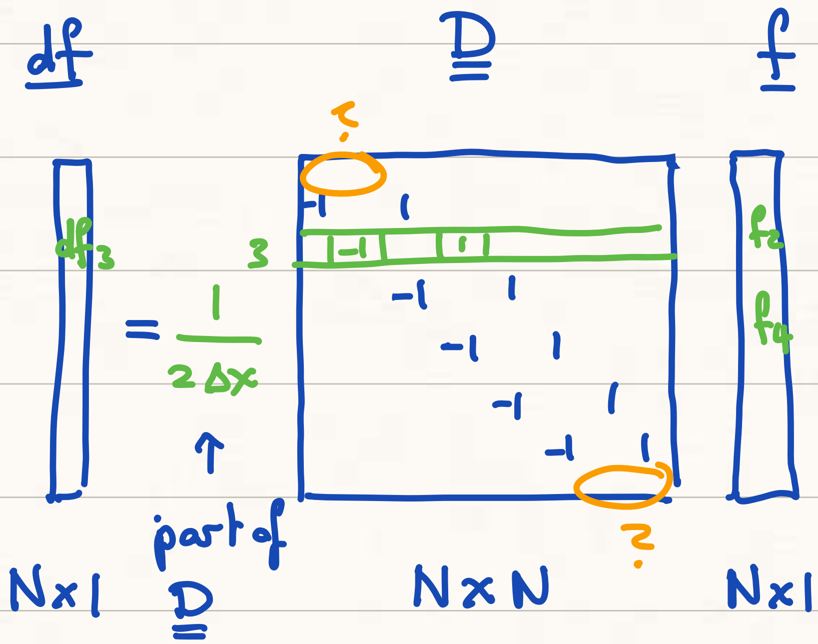
What is discrete ~~equi~~ approx. of $\mathcal{D}$?

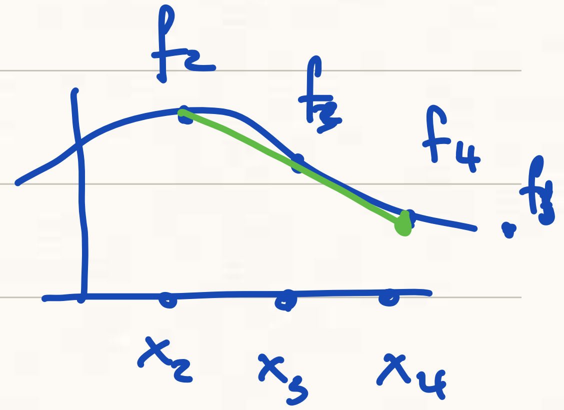
$$\underline{df} = \underline{\underline{D}}\;\underline{f}$$

⇒ has to be a matrix because its linear and relates two vectors.

Note: all our vectors are column vectors

$\Rightarrow$ Differentiation matrix $\underline{\underline{D}}$





Central difference

$$df_3 = \frac{f_4 - f_2}{2\Delta x}$$

$$df_4 = \frac{f_5 - f_3}{2\Delta x}$$

$\Rightarrow$ $\underline{\underline{D}}$ simple bi-diagonal structure

Note: Boundaries require special treatment

What about 2^{nd} derivatives? $\qquad \frac{d^2 f}{dx^2} = \ddot{f}$

$$\underline{ddf} = \underline{\underline{D}}\,\underline{df} = \underline{\underline{D}}\,\underline{\underline{D}}\,\underline{f} = \underline{\underline{D}}^2\,\underline{f}$$

$\Rightarrow$ extended to arbitrarily high derivatives
but every time we multiply by $\underline{\underline{D}}$ we
incur an error!

High derivatives $\Rightarrow$ higher order discretizations