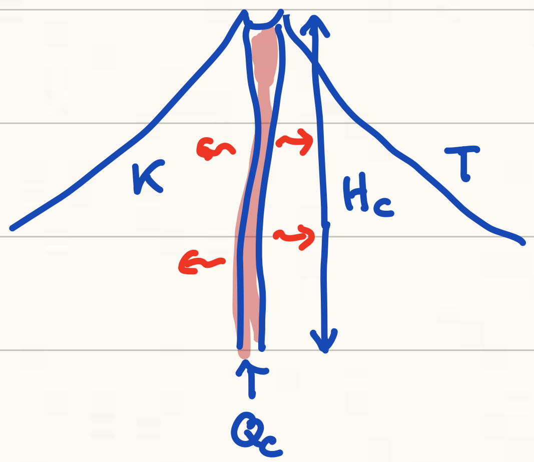
Lecture 3: Conservative Finite Differences

Aim: ⇒ motivate our choice of FD

Example problem:

Steady heat flow from volcanic conduit

(Flow from well → hydrologic equivalent)



H_c = height of conduit

κ = thermal cond. of rock

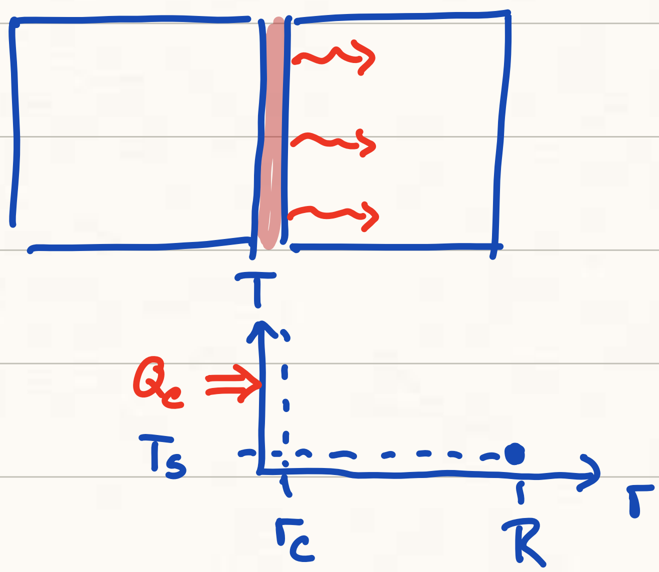
T_s = surf. temp

Q_c = heat flow

r_c = conduit radius

R = volcano radius

1 D simplification



Heat equation

$$\cancel{\rho c_p \frac{\partial T}{\partial t}}^{\,0} - \nabla\cdot\kappa\nabla T = 0$$

$$\cancel{-}\nabla\cdot\cancel{\kappa}\nabla T = 0 \qquad -1/\kappa$$

$$\Rightarrow \quad \nabla\cdot\nabla T = 0$$

$$\cancel{\frac{1}{r}}\frac{\partial}{\partial r}\, r\, \frac{\partial T}{\partial r} = 0 \qquad \cdot r$$

PDE: $\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = 0 \qquad r \in [r_c, R]$

Boundary conditions:

1) $T(R) = T_s$ — fix T at surface (Dirichlet BC)

2) $Q_c = A_c\, q(r_c)$ — $A_c = 2\pi r_c H_c$

$q = -\kappa \nabla T = -\kappa \frac{dT}{dr}$

$$Q_c = -A_c \kappa \left.\frac{\partial T}{\partial r}\right|_{r_c}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r_c} = -\frac{Q_c}{A_c \kappa}$$

fix $\frac{\partial T}{\partial t}$ at conduit (heat flow)

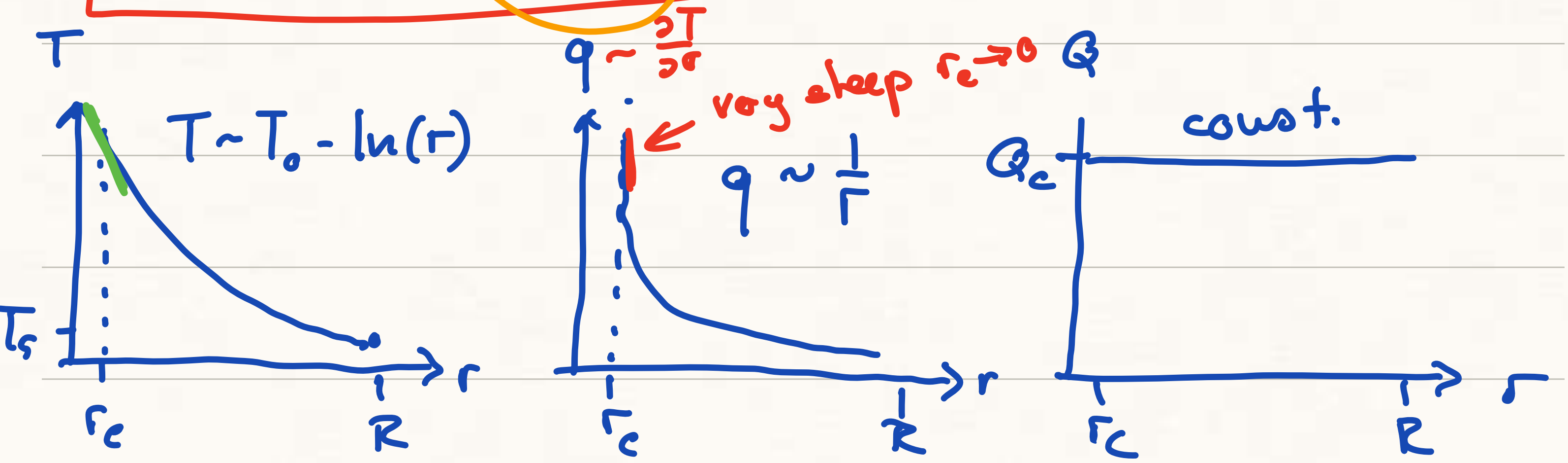
(Neumann BC)

Summary:

PDE: $\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) = 0 \qquad r \in [r_c, R]$

BC: $T(R) = T_s$

$$\left.\frac{\partial T}{\partial r}\right|_{r_c} = -\frac{Q_c}{A_c \kappa}$$



Finite difference discretization

PDE: $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = r\frac{d^2T}{dr^2} + \frac{dT}{dr} = 0$

Look up Stand. FD: $\frac{dT}{dr} \approx \frac{T_{i+1} - T_{i-1}}{2\Delta r}$

$$\frac{d^2T}{dr^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2}$$

Finite differencing in conservation form

$$-\nabla\cdot[\kappa\nabla T]=0$$

radial: $-\frac{1}{r}\frac{d}{dr}\Big(\underbrace{r\kappa\frac{dT}{dr}}_{\text{not simple deriv.}}\Big)=0$

chain rule $\rightarrow$ have simple derivatives

For now consider 1D linear geometry.

$$-\nabla\cdot[\kappa\nabla T]=f_s$$

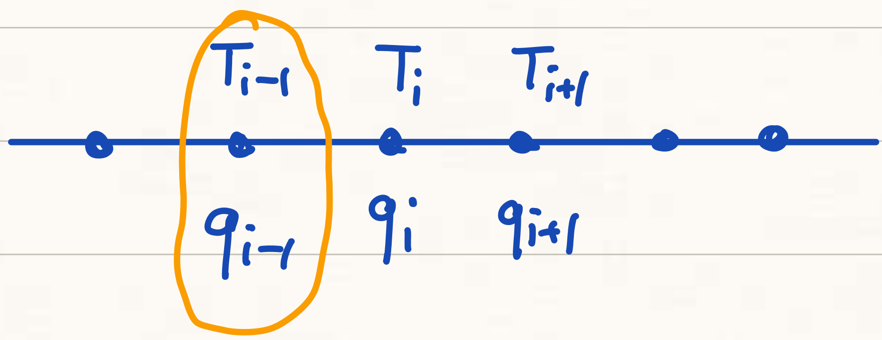
$$-\frac{d}{dx}\left[\kappa\frac{dT}{dx}\right]=f_s$$

Rewrite as div-grad system

1) $\nabla\cdot q = f_s \overset{1D}{\rightarrow} \frac{dq}{dx}=f_s \overset{FD}{\Rightarrow} \frac{q_{i+1}-q_{i-1}}{2\Delta x}=f_i$

2) $q=-\kappa\nabla T \overset{1D}{\rightarrow} q=-\kappa\frac{dT}{dx} \qquad q_i=-\kappa_i\frac{T_{i+1}-T_{i-1}}{2\Delta x}$

notice: 2 variables (q,T) colocated on grid



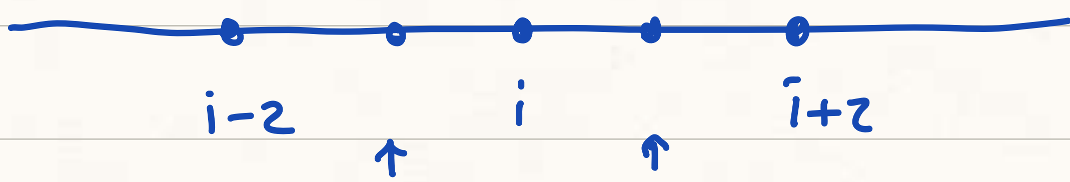
Either solve for both T & q simultaneously or eliminate q by substituting $2 \to 1$

$$\frac{q_{i+1}-q_{i-1}}{2\Delta x} = f_i$$

$$\frac{1}{2\Delta x}\left[\underbrace{-k_{i+1}\frac{T_{i+2}-T_i}{2\Delta x}}_{q_{i+1}} + \underbrace{k_{i-1}\frac{T_i-T_{i-2}}{2\Delta x}}_{q_{i-1}}\right] = f_i$$

$$\frac{1}{4\Delta x}\left[-k_{i+1}T_{i+2} + (k_{i+1}-k_{i-1})T_i - k_{i-1}T_{i-2}\right] = f_i$$

↪ Matlab



⇒ wide stencil ⇒ decoupling of even & odd nodes

⇒ oscillations

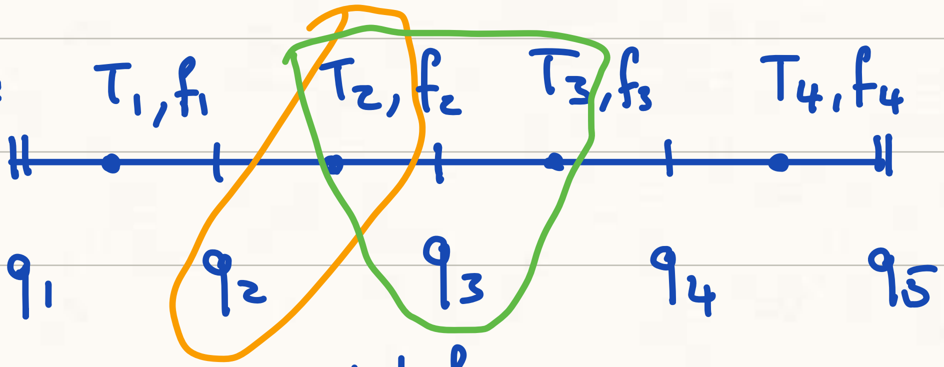
but solution is improved in some ways.

Conservative Finite Differences / Finite Volumes

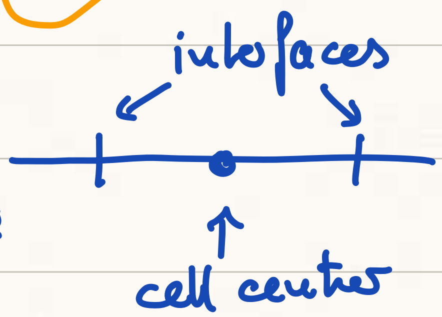
Aims: 1) compact stencil } staggered grid
2) no decoupling

scalars:

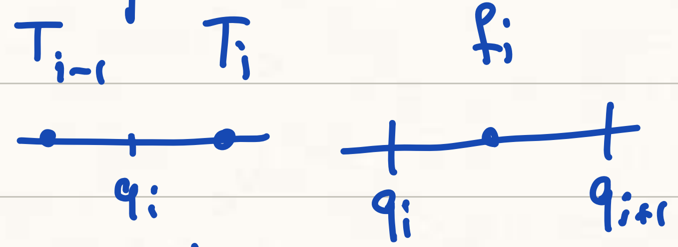
fluxes:



cell control volume:



q & T not co-located



$\Rightarrow$ Discretize div-grad system:

1) $\nabla \cdot q = f_s \xrightarrow{1D} \frac{dq}{dx} = f \xrightarrow{CFD} \frac{q_{i+1} - q_i}{\Delta x} = f_i$

2) $q = -k \nabla T \xrightarrow{1D} q = -k \frac{dT}{dx} \xrightarrow{CFD} q_i = -k \frac{T_{i+1} - T_i}{\Delta x}$

for now k is constant

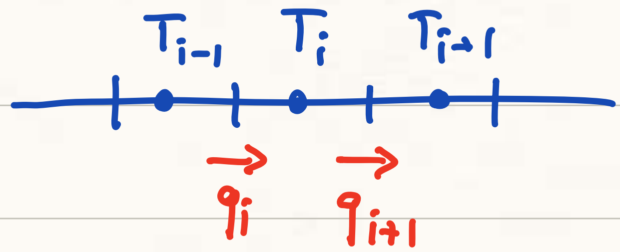
eliminate q_i by substituting ② into ①

Note: Indices are wrong here → new page

$$\frac{q_{i+1} - q_i}{\Delta x} = f_i$$

$$\frac{1}{\Delta x}\left[-k\,\frac{T_{i+2} - T_{i+1}}{\Delta x} + k\,\frac{T_{i+1} - T_i}{\Delta x}\right] = f_i$$

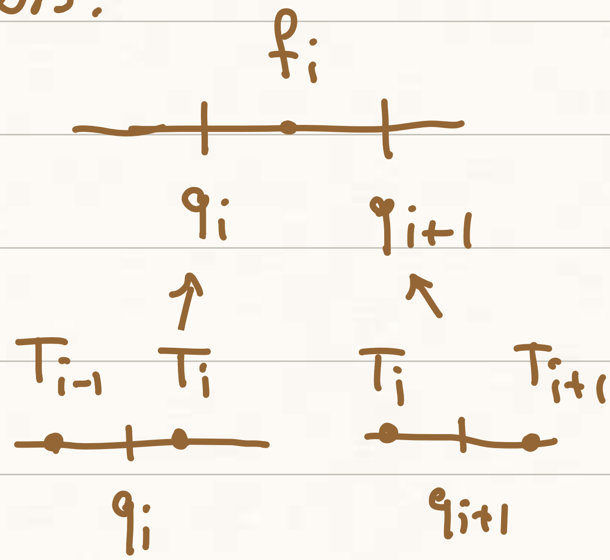
$$\Rightarrow \; -\frac{k}{\Delta x^2}\left[T_{i+1} - 2T_i + T_{i-1}\right] = f_i$$



$\Rightarrow$ excellent solution because heat is conserved exactly

"Discrete conservation"

Indices fixed after class:



$$\frac{q_{i+1} - q_i}{\Delta x} = f_i$$

$$q_i = -k \frac{T_i - T_{i-1}}{\Delta x}$$

$$q_{i+1} = -k \frac{T_{i+1} - T_i}{\Delta x}$$

substitute q_i & q_{i+1} into $\frac{q_{i+1} - q_i}{\Delta x} = f_i$

$$\frac{1}{\Delta x}\left[-k \frac{T_{i+1} - T_i}{\Delta x} - \left(-k \frac{T_i - T_{i-1}}{\Delta x}\right)\right] = f_i$$

$$-\frac{k}{\Delta x^2}\left[T_{i+1} - 2T_i + T_{i-1}\right] = f_i$$