# Linearize discrete equation

$$\underline{r}(\underline{u}) = -\underline{\underline{D}}\ \underline{\underline{Kd}}(\underline{u})\ \underline{\underline{G}}\ \underline{u} - \underline{fs}$$

Need to differentiate $\underline{\underline{Kd}}$ = spdiags$(\underline{\underline{M}}\ \underline{\kappa}(\underline{u}), 0, Nfx, Nfx)$

short hand: $\underline{\underline{Kd}}(u) = \underline{\underline{I}}_f(\underline{\underline{M}}\ \underline{\kappa}(\underline{u}))$

$\underline{\underline{I}}_f$ is a Nfx by Nfx matrix with

$\underline{\text{Kmean}} = \underline{\underline{M}}\ \underline{\kappa}(\underline{u})$ on diagonal

Linearize around $\underline{u}^k$ (dropping superscript)

$$\frac{d}{d\varepsilon}\underline{r}(\underline{u} + \varepsilon\,\underline{du})\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Big\langle -\underline{\underline{D}}\Big[\underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du})\ \underline{\underline{G}}\ (\underline{u}+\varepsilon\,\underline{du})\Big]\Big\rangle_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\Big\langle -\underline{\underline{D}}\Big[\underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du})\ \underline{\underline{G}}\ \underline{u} + \varepsilon\,\underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du})\ \underline{\underline{G}}\ \underline{du}\Big]\Big\rangle_{\varepsilon=0}$$

$$= -\underline{\underline{D}}\Big[\frac{d}{d\varepsilon}\underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du})\ \underline{\underline{G}}\ \underline{u} + \underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du})\ \underline{\underline{G}}\ \underline{du} + \cancelto{0}{\varepsilon}\frac{d}{d\varepsilon}\underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du})\ \underline{\underline{G}}\ \underline{du}\Big]$$

$$\frac{d}{d\varepsilon}\underline{\underline{Kd}}(\underline{u}+\varepsilon\,\underline{du}) = \frac{d}{d\varepsilon}\underline{\underline{I}}_f\big(\underline{\underline{M}}\ \underline{\kappa}(\underline{u}+\varepsilon\,\underline{du})\big) = \underline{\underline{I}}_f\Big(\underline{\underline{M}}\ \frac{d}{d\varepsilon}\underline{\kappa}(\underline{u}+\varepsilon\,\underline{du})\Big)$$

$$= \underline{\underline{I}}_f\Big(\underline{\underline{M}}\ \frac{d\kappa}{du}\Big|_{\bar{u}}\ \underline{du}\Big)$$

?

Need to be careful! Both $\left.\frac{d\kappa}{du}\right|_{\underline{u}}$ and $\underline{du}$ are Nx by 1 vectors that are multiplied point wise:

$$\left.\frac{d\kappa}{du}\right|_{\underline{u}} .* \underline{du} = \underline{\underline{I}}_c\left(\left.\frac{d\kappa}{du}\right|_{\bar{u}}\right) * \underline{du}$$

$$\Rightarrow \frac{d}{d\varepsilon}\underline{\underline{Kd}}(\underline{u}+\varepsilon\underline{du}) = \underline{\underline{I}}_f\left(\underline{\underline{M}}\;\underline{\underline{I}}_c\left(\left.\frac{d\kappa}{du}\right|_{\bar{u}}\right)\underline{du}\right)$$

return to linearization

$$\frac{d}{d\varepsilon}\underline{r}(\underline{u}+\varepsilon\underline{du})\Big|_{\varepsilon=0} = -\underline{\underline{D}}\left[\underbrace{\frac{d}{d\varepsilon}\underline{\underline{Kd}}(\underline{u}+\varepsilon\underline{du})\,\underline{\underline{G}}\,\underline{u}}_{\text{focus on this term}} + \underline{\underline{Kd}}(\underline{u}+\varepsilon\underline{du})\,\underline{\underline{G}}\,\underline{du}\right]_{\varepsilon=0}$$

$$\underline{\underline{I}}_f\left(\underbrace{\underline{\underline{M}}\;\underline{\underline{I}}_c\left(\left.\frac{d\kappa}{du}\right|_{\bar{u}}\right)\underline{du}}_{Nfx\cdot 1}\right)\;\underbrace{\underline{\underline{G}}\,\underline{u}}_{Nfx\cdot 1}$$

note that $\underline{du}$ "update" is our unknown

we need to be able to extract it $\rightarrow$ $\underline{\underline{L}}\,\underline{du} =$

again we are multiplying two vectors elementwise

$$\underline{a} .* \underline{b} = \underline{b} .* \underline{a} \qquad \text{commutes}$$

$$\underline{\underline{I}}(\underline{a})\,\underline{b} = \underline{\underline{I}}(\underline{b})\,\underline{a}$$

$$\Rightarrow \underline{\underline{I}}_f\Big(\underbrace{\underline{\underline{M}}\,\underline{\underline{I}}_c\Big(\frac{d\kappa}{du}\Big|_{\bar{u}}\Big)\underline{du}}_{\underline{a}}\Big)\,\underbrace{\underline{\underline{G}}\,\underline{u}}_{\underline{b}} = \underline{\underline{I}}_f(\underbrace{\underline{\underline{G}}\,\underline{u}}_{\underline{b}})\,\underbrace{\underline{\underline{M}}\,\underline{\underline{I}}_c\Big(\frac{d\kappa}{du}\Big|_{\bar{u}}\Big)\underline{du}}_{\underline{a}}$$

substitute

$$\frac{d}{d\varepsilon}\underline{r}(\underline{u}+\varepsilon\underline{du})\Big|_{\varepsilon=0} = -\underline{\underline{D}}\Big[\underbrace{\underline{\underline{I}}_f(\underline{\underline{G}}\,\underline{u})}_{\underline{\underline{GU}}}\,\underbrace{\underline{\underline{M}}\,\underline{\underline{I}}_c\Big(\frac{d\kappa}{du}\Big|_{\underline{u}}\Big)}_{\underline{\underline{dKd}}}\underline{du} + \underline{\underline{Kd}}(\underline{u})\,\underline{\underline{G}}\,\underline{du}\Big]$$

$$= \underbrace{-\underline{\underline{D}}\big[\underline{\underline{GU}}\,\underline{\underline{dKd}} + \underline{\underline{Kd}}\,\underline{\underline{G}}\big]}_{\underline{\underline{J}}}\,\underline{du}$$

$$\boxed{\begin{aligned} \underline{\underline{J}} &= -\underline{\underline{D}}\big[\underline{\underline{GU}}\,\underline{\underline{dKd}} + \underline{\underline{Kd}}\,\underline{\underline{G}}\big] \\ J &= -\nabla\Big(\frac{d\kappa}{du}\Big|_{\bar{u}}\nabla\bar{u} + \kappa(\bar{u})\nabla\Big] \end{aligned}}$$

discrete

continuous

$$\underline{\underline{GU}} \leftrightarrow \nabla\bar{u}\,, \quad \underline{\underline{Kd}} \leftrightarrow \kappa(\underline{u})\,, \quad \underline{\underline{dKd}} \leftrightarrow \frac{d\kappa}{du}\Big|_{\bar{u}}$$

Note: $\underline{\underline{GU}}$, $\underline{\underline{Kd}}$ and $\underline{\underline{dKd}}$ are all funcions of $\underline{u}$