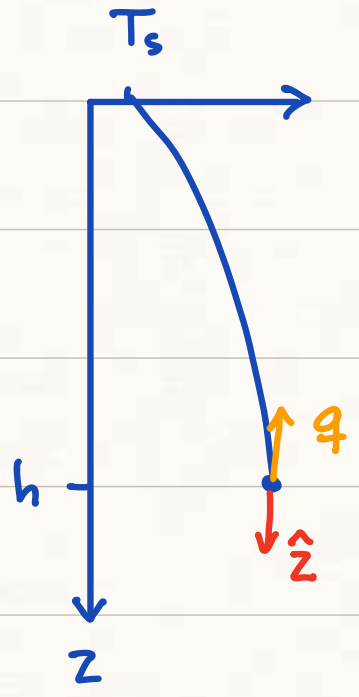
Geotherm with mantle heat flow

Steady geotherm:

PDE: $-\nabla \cdot k \nabla T = \rho H_0 e^{\frac{-z}{h_r}}$

BC: $T(0) = T_s$ $\quad q \cdot \hat{z}|_h = -|q_m|$



Typically flux/Neumann BC uses outward normal.

$\Rightarrow$ inflow is negative number

$|q_m|$ magnitude of mantle heat flow

$\Rightarrow$ makes minus sign explicit.

This is a bit akward $\Rightarrow$ we use inward normal!

Neumann Boundary Conditions

Dirichlet BC's prescribe the unknown on boundary, so that it can be eliminate it. Neumann BC's prescribe the flux / derivative, so that we still have to solve for the unknown on boundary.

⇒ Neumann BC's are not implemented as constraints

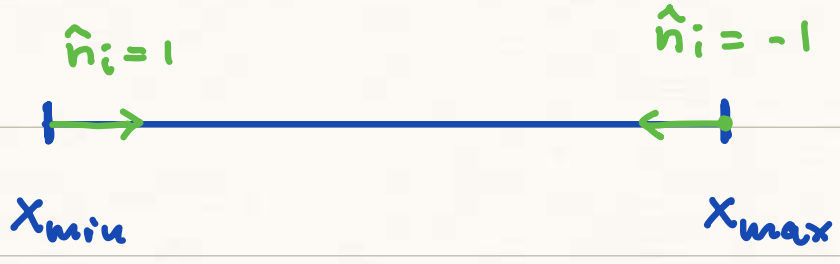
Sign convention

In this class we consider inflows positive for reasons that will become clear in a minute.

$$q \cdot \hat{n}_i = q_B$$

$\hat{n}_i$ = inward normal

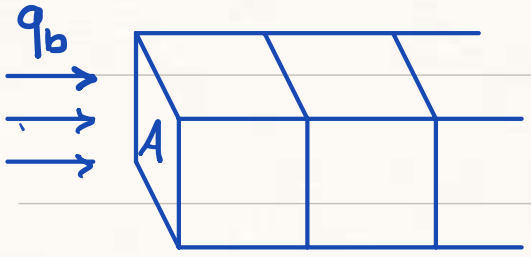
q_B = bnd flux



⇒ $q_B > 0$ · is an inflow

Implementation of Neumann BC

We implement flux BC as an equivalent source / sink term to ensure conservation.



face area

Total flow rate across bnd face: $Q_b = A\, q_b$

Equivalent source term: $Q_b = V f_n$

cell volume

$$\Rightarrow \boxed{f_n = q_b \frac{A}{V}} \quad \text{(for a single cell)}$$

Note: sign of f_n is automatically correct because $q_b > 0$ is an inflow and f_n has same sign.

In general $\underline{f_n}$ is Nx by 1 r.h.s. vector with $\underline{Nn}$ non-zero entries, one for each Neumann BC applied.

For a problem with Neumann BC's the linear system is: $\underline{\underline{L}}\,\underline{h} = \underline{f_s} + \underline{f_n}$

To construct fn we define:

BC.dof_neu = Nn by 1 vector of cells with Neumann BC

BC.dof_f_neu = Nn by 1 vector of faces with Neum. BC

BC.qb = Nn by 1 vector of prescribed fluxes

and add cell volumes and face areas to Grid.

Grid.A = Nf by 1
Grid.V = N by 1
} assume other dimensions are unity!

Compute and place the Nn entries of fn

$$fn(BC.dof\_f\_neu) = \underline{qb} * Grid.A(BC.dof\_f\_neu)/Grid.V(BC.dof\_neu)$$

⇒ Neumann BC can be implemented in one line in build_bnd.m!