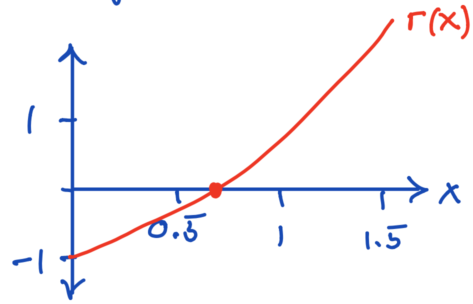
Newton-Raphson Method

Suppose we have the simple non-linear function

$r(x) = e^x - 2$



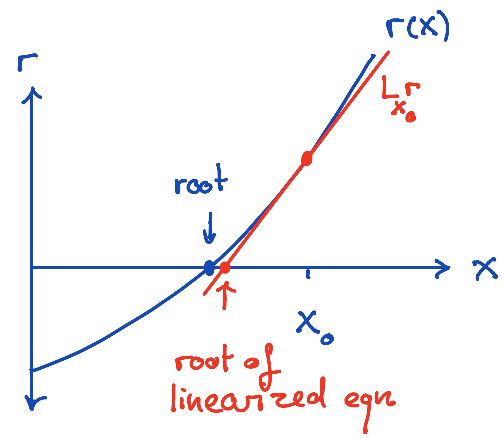
and we want to find its root, $r(x) = 0$.

$e^x = 2 \quad \Rightarrow \quad x = \log 2 = 0.6931$

Here we can solve the equation analytically, but in general the root has to be found numerically by iteration.

Iteration means a sequence of improving approximations that starts from an initial guess, x_0.

If the initial guess is close enough to the root we can linearize the function with Taylor series and find the root of the linearized equation.



$$L_{x_0} r = r(x_0) + \left.\frac{dr}{dx}\right|_{x_0} \underbrace{(x - x_0)}_{\Delta x} + O(\Delta x^2) \qquad \text{Taylor series}$$

Root of $L_{x_0} r$: $r(x_0) + \left.\frac{dr}{dx}\right|_{x_0} \Delta x = 0$

$$\Rightarrow \quad \Delta x = -r(x_0) \Big/ \left.\frac{dr}{dx}\right|_{x_0}$$

$$\text{root}: \quad x_1 = x_0 + \Delta x$$

Newton-Raphson method turns this into an iterative procedure that converges to the root of $r(x)$ quadratically.

at k-th iteration: $\boxed{\begin{aligned} \Delta x^k &= -r(x^k) \Big/ \left.\frac{dr}{dx}\right|_{x^k} \\ x^{k+1} &= x^k + \Delta x^k \end{aligned}}$ Newton-Raphson single equation

→ look at demo_scalar_newton.m

Newton-Raphson for systems of equations

The same approach works for a system of N non-linear algebraic equations. $\underline{r}(\underline{u}) = 0$

$\underline{r}$ is N by 1 vector of equation

$\underline{u}$ is N by 1 vector of unknowns

Again we use Taylor expansion to linearize the system at $\underline{u}^k$ and solve for update $\underline{\Delta u}^k$

$$L_{\underline{u}^k}\underline{r} = \underline{r}(\underline{u}^k) + \underline{\Delta u}^k \frac{d\underline{r}(\underline{u})}{d\underline{u}} + \mathcal{O}(|\underline{\Delta u}^2|) \qquad \text{lin. of } \underline{r} \text{ at } \underline{u}^k$$

Here the derivatives of the N residuals with respect to the N unknowns form the N by N Jacobian matrix.

$$\boxed{\underline{\underline{J}}(u) = \frac{d\underline{r}(u)}{\underline{u}}}$$

We determine the unknown update $\underline{\Delta u}^k$ by finding the root of the linearized equation $L_{\underline{u}^k}\underline{r} = 0$

$$\boxed{\begin{aligned} \underline{\Delta u}^k &= -\underline{\underline{J}}(\underline{u}^k)^{-1}\, \underline{r}(\underline{u}^k) \\ \underline{u}^{k+1} &= \underline{u}^k + \underline{\Delta u}^k \end{aligned}}$$

Newton-Raphson for system of equations

Note: - Need to find Jacobian matrix $\underline{\underline{J}}$

- Need to solve a linear system every iteration
- $\underline{\underline{J}}(\underline{u}^k) \Rightarrow \underline{\underline{J}}$ needs to be recomputed every iteration
- Remember this is not guaranteed to converge!