# Scaling the Navier-Stokes Equations

Last time we derived the Navier Stokes Equations

$$\rho \frac{\partial \underline{v}}{\partial t} + \nabla \cdot [\rho(\underline{v} \otimes \underline{v}) - \mu(\nabla \underline{v} + \nabla^T \underline{v})] = -\nabla p + \rho \underline{g} \quad \& \quad \nabla \cdot \underline{v} = 0$$

As long as we don't consider buoyancy driven flows we can absorb the gravity term into a reduced pressure.

$$-\nabla p + \rho g = -\nabla p - \rho g \hat{z} = -\nabla p - \rho g \nabla z = -\nabla(p + \rho g z) = -\nabla \pi$$

where $\pi = p + \rho g z$ is the reduced pressure.

This can be related to the hydraulic head $h = \frac{\pi}{\rho g}$ in case you are used to thinking in heads.

Here we leave the NS-eqn in terms of reduced pressure

$$\boxed{\rho \frac{\partial \underline{v}}{\partial t} + \nabla \cdot [\rho(\underline{v} \otimes \underline{v}) - \mu(\nabla \underline{v} + \nabla^T \underline{v})] = -\nabla \pi}$$

First we non-dimensionalize with generic scales to define the standard dimensionless parameters then specialize it to particular cases.

Let's take stock of parameters & variables:

- Dependent variables: $\underline{v}$, $\pi$ ($p$)
- Independent variables: $\underline{x}$, $t$
- Parameters: $\rho$, $\mu$ + (geometry, BC's, IC)

We scale variables and parameters functions of the variables

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \mu' = \frac{\mu}{\mu_c}$$

$$\Rightarrow \quad \frac{\rho v_c}{t_c} \frac{\partial \underline{v}'}{\partial t'} + \nabla' \cdot \left[\frac{\rho v_c^2}{x_c}(\underline{v}' \otimes \underline{v}') - \frac{\mu_c v_c}{x_c^2} \mu' (\nabla' \underline{v}' + \nabla'^T \underline{v}')\right] = -\frac{\pi_c}{x_c} \nabla' \pi'$$

First, we will re arrange the char. groups into a standard advection diffusion scaling by setting the accumulation term to unity. After dropping the primes we have

$$\frac{\partial \underline{v}}{\partial t} + \nabla \cdot \left[ \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1} (\underline{v} \otimes \underline{v}) - \underbrace{\frac{\mu_c t_c}{\rho x_c^2}}_{\Pi_2} \mu' (\nabla \underline{v} + \nabla^T \underline{v}) \right] = \underbrace{\frac{\pi_c t_c}{\rho v_c x_c}}_{\Pi_3} \nabla \pi$$

So that we have 3 dimensionless groups. The first two are identical to those arising in the standard advection diffusion equation. They define advective and diffusive timescales.

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow$ advective time scale: $t_A = \frac{x_c}{v_c}$

"time for fluid to flow distance $x_c$"

$\Pi_2 = \frac{\mu_c t_c}{\rho x_c^2} = \frac{\nu_c t_c}{x_c^2} \Rightarrow$ diffusive time scale: $t_D = \frac{x_c^2}{\nu_c}$

where $\nu_c = \frac{\mu_c}{\rho}$ is the momentum diffusivity

"time for momentum/vorticity to diffuse distance $x_c$"

In our applications $\Pi_3 = 1$ will be used to define an internal pressure scale $\pi_c = \rho v_c x_c / t_c$. But in pressure driven flows, e.g. pipe flows, it could also define a pressure-based time scale $t_p = \rho v_c x_c / \pi_c$ where $\pi_c$ would be an external pressure scale.

Choosing a diffusive timescale $t_c = t_D$ we have

$$\frac{\partial \underline{v}}{\partial t} + \nabla \cdot \left[ \frac{v_c x_c}{\nu} (\underline{v} \otimes \underline{v}) - \mu (\nabla \underline{v} + \nabla^T \underline{v}) \right] = \nabla \pi$$

Hence we have one dimensionless group in the form of a Peclet number comparing advection & diffusion of momentum

$$\boxed{Pe_m = \frac{v_c x_c}{\nu_c} = \frac{\rho v_c x_c}{\mu_c} = \frac{t_D}{t_A} = Re}$$ Reynolds Number

In fluid mechanics this dimensionless number is called the Reynolds number.

$$\boxed{\frac{\partial \underline{v}}{\partial t} + \nabla \cdot \left[ Re (\underline{v} \otimes \underline{v}) - \mu (\nabla \underline{v} + \nabla^T \underline{v}) \right] = \nabla \pi}$$

Clearly, the momentum term vanishes if $Re \to 0$

For our application in viscous flow of ice, ie, glaciers & ice sheets we have the following parameters:

$\rho = 10^3$

$v_c = 1 - 100 \ m/yr \approx 10^{-7} - 10^{-5} \ \frac{m}{s} \sim 10^{-6} \ \frac{m}{s}$ $(yr \sim 10^7 s)$

$\mu_c = 10^{13} - 10^{15}$ Pas $\sim 10^{14}$ Pas

$x_c = 10^2 - 10^3 \ m \sim 10^2 \ m$

$$Re = \frac{\rho v_c x_c}{\mu_c} = 10^{3-6+2-14} = 10^{-15}$$

$\Rightarrow$ advective momentum transport is neglicible

But is it worth resolving diffusive time scales?

$$t_D = \frac{x_c^2 \rho}{\mu_c} = 10^{4+3-14} \ s = 10^{-7} \ s$$

Very short time scale compared to years to 100 years of glacier response

$\Rightarrow$ Choose a different non-dimensionalization

- clearly the viscosity term is important
- like to get rid of time derivative

$\Rightarrow$ scale to viscous term

$$\frac{\rho v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \nabla'\cdot\left[\frac{\rho v_c^2}{x_c}(\underline{v}'\otimes\underline{v}') - \underset{\substack{\uparrow \\ \text{divide by this to set viscous term to unity}}}{\frac{\mu_c v_c}{x_c^2}}\mu'(\nabla'\underline{v}' + \nabla'^T\underline{v}')\right] = -\frac{\pi_c}{x_c}\nabla'\pi'$$

dropping primes:

$$\frac{\rho x_c^2}{\mu_c t_c}\frac{\partial \underline{v}}{\partial t} + \nabla\cdot\left[\underbrace{\frac{v_c x_c}{\nu}}_{Re}(\underline{v}\otimes\underline{v}) - \mu(\nabla v + \nabla^T v)\right] = -\underbrace{\frac{\pi_c x_c}{\mu_c v_c}}_{=1}\nabla\pi_c \quad \Rightarrow \pi_c = \frac{\mu_c v_c}{x_c}$$

choose advective time scale: $t_c = \frac{x_c}{v_c}$

$$Re\left(\frac{\partial \underline{v}}{\partial t} + \nabla\cdot(\underline{v}\otimes\underline{v})\right) - \nabla\cdot\left[\mu(\nabla\underline{v} + \nabla^T\underline{v})\right] = -\nabla\pi$$

limit $Re \ll 1$ we obtain <u>Stokes</u> equation

$$\boxed{\begin{aligned}\nabla\cdot\left[\mu(\nabla\underline{v} + \nabla^T\underline{v})\right] &= \nabla\pi \\ \nabla\cdot\underline{v} &= 0\end{aligned}}$$

Here written for variable viscosity

In the limit of constant viscosity $\mu' = 1$

$$\boxed{\begin{aligned}\nabla^2\underline{v} &= \nabla\pi \\ \nabla\cdot\underline{v} &= 0\end{aligned}}$$

$\Rightarrow$ Stokes/viscous flow is steady, ie. flow field instantaneously adjustst to changes in any forcing or BC.

# Boundary conditions for Stokes Equation

1) No slip condition at solid boundary


is boundary is stationary $\underline{v}_b = 0$

$$\boxed{\underline{v}\big|_{\underline{x}\in\partial\Omega} = \underline{0}}$$

Dirichlet BC

$\Rightarrow$ prescribe velocity

$\Rightarrow$ implemented with constraints

2) Free slip / No shear stress condition

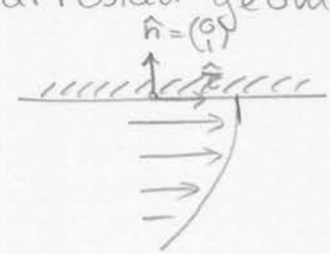

$\underline{v}\cdot\hat{n}\big|_{\partial\Omega} = 0$ no flow across boundary

$\hat{\tau}\cdot(\underline{\underline{\sigma}}\cdot\hat{n})\big|_{\partial\Omega} = 0$ no shear stress

here $\underline{t} = \underline{\underline{\sigma}}\cdot\underline{n}$ is the traction vector on boundary

$t_{\parallel} = \hat{\underline{\tau}}\cdot\underline{t}$ component of traction parallel to boundary (shear stress)

In cartesian geometry: $\hat{n} = \begin{pmatrix}0\\1\end{pmatrix}$ $\hat{\tau} = \begin{pmatrix}1\\0\end{pmatrix}$




$$\hat{\underline{\tau}}\cdot(\underline{\underline{\sigma}}\cdot\hat{\underline{n}}) = (1 \quad 0)\begin{pmatrix} v_{x,x} & \frac{1}{2}(v_{x,y}+v_{y,x}) \\ \frac{1}{2}(v_{x,y}+v_{y,x}) & v_{y,y}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}(v_{x,y}+v_{y,x})$$

since $v_y = 0$ along bnd $\frac{\partial v_y}{\partial x} = 0$

$$\Rightarrow \quad \frac{\partial v_x}{\partial y}\Big|_{\partial\Omega} = 0 \qquad \text{Neumann BC}$$

more generally $\boxed{\nabla(\underline{v}\cdot\hat{\tau})\cdot n = 0}$

Note: We don't need to impose BC's on the (reduced) pressure!
This is not trivially obvious, but can be demonstrated by showing that the pressure is a Lagrange multiplyer that enforces incompressibility.