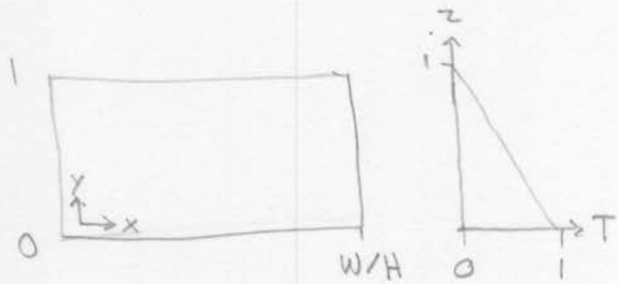
Modeling Stokes Convection

Governing Equations

$$\boxed{\begin{aligned} &1)\quad \nabla\cdot[\mu(\nabla\underline{v} + \nabla\underline{v}^T)] - \nabla\pi = T\,\hat{z} \\ &2)\quad \nabla\cdot\underline{v} = 0 \\ &3)\quad \frac{\partial T}{\partial t} + \nabla\cdot\left[\underline{v}T - \frac{1}{Ra}\nabla T\right] = 0 \end{aligned}}$$



Estimates of Rayleigh number in ice: $Ra = \dfrac{\Delta\rho\, g\, H^3}{\mu_c\, \kappa}$

$$\left.\begin{aligned} \mu_c &= 10^{14} \quad Pa\,s \\ g &= 10^{0} \quad m/s^2 \\ \Delta\rho &= 10^{1} \quad kg/m^3 \\ k &= 10^{-6} \quad m^2/s \\ H &= 10^{4} \quad m^2 \end{aligned}\right\} \qquad \begin{aligned} Ra &= 10^1 \cdot 10^0 \cdot (10^{4})^3 \cdot 10^{-14} \cdot 10^{6} \\ &= 10^{1+12-14+6} = 10^5 \end{aligned}$$

$\Rightarrow$ Heat transport is expected to be dominated by advection.

Implementation of the convection code

Need to solve three coupled equations.

1) Flow problem: solve 1 & 2 given T

2) Transport problem: solve 3 given $\underline{v}$

Note, that Transport problem is solved on same grid as pressure $\Rightarrow$ use G_p & D_p for transport.

Note, our problem has only one parameter, Ra.

General code outline

1) Define physical & numerical parameters

   Ra

   Θ, t_{max}, dt_max, cfl, plot/save interval

2) Build grid & ops
   - general operators → build_stokes_grid/ops
   - Stokes operator
   - ADE operators *
   - Boundary conditions.

3) Transient evolution
   - define initial condition

     T, time, counters (i, k)
   - while loop time < tmax
     - Flow problem: 1) Buoyancy term from T *
       2) solve stokes
     - Transport problem: 1) Determine time step *
       2) solve transport.
     - Plot & save solution:

The following items need discussion:

1) ADE operator
2) Buoyancy term
3) Time step selection

1) Advection-diffusion equation

$$\frac{\partial T}{\partial t} + \boxed{\nabla \cdot [\underline{v} T - \frac{1}{Ra} \nabla T]} = 0$$

since $\underline{u} = [\underline{v}\ \underline{p}]$ in stokes problem well treat convection and use $\underline{T}$ as solution vector.

We'll use the standard θ-method for time step discretization

$$\underline{\underline{I}}\ \frac{\underline{T}^{n+1} - \underline{T}^{n}}{\Delta t} + \underline{\underline{Dp}} * [\underline{\underline{A}}(\underline{v}) - \frac{1}{Ra} \underline{\underline{Gp}}] (\theta \underline{T}^{n} + (1-\theta) \underline{T}^{n+1}) = 0$$

Collect $\underline{T}^{n+1}$ on lhs and $\underline{T}^{n}$ on the rhs.

$$\left\langle \underline{\underline{I}} + (1-\theta) \Delta t \underbrace{\underline{\underline{Dp}} * [\underline{\underline{A}}(\underline{v}) - \frac{1}{Ra} \underline{\underline{Gp}}]}_{\text{L\_trans}} \right\rangle \underline{T}^{n+1} = \left\langle \underline{\underline{I}} - \theta \Delta t \underbrace{\underline{\underline{Dp}} * [\underline{\underline{A}}(\underline{v}) - \frac{1}{Ra} \underline{\underline{Gp}}]}_{\text{L\_trans}} \right\rangle \underline{T}^{n}$$

Hence we have: $\quad$ Im_trans = I_trans + $(1-\theta)\Delta t *$ L_trans

Ex_trans = I_trans $-$ $\theta \Delta t *$ L_trans

$\uparrow$ just emphasizing this is the identity on p-grid.

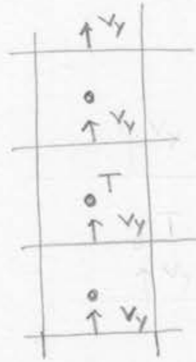
Note: These operators change each timestep because the flow field changes $\Rightarrow$ anonymous functions

2) Buoyancy term in Stokes Equation

The new term in stokes equation

$$\nabla\cdot[\mu(\nabla \underline{v} + \nabla^T \underline{v})] - \nabla \pi = T\,\hat{z}$$

buoyancy term



T is a variable that lives in cell centers but here it is used as body force on the v_z (or v_y) velocity.

Interpolate T to the cell faces
⇒ use comp-mean (with $p=1$ for arithmetic average)
This will give us a diagonal matrix.
Need to extract the diagonal from matrix and then only add the averages on y-faces to rhs.

3) Timestep selection

To minimize numerical diffusion, we are explicitly time stepping the advection-diffusion equation. ($\theta = 1$)
Conditional stability requires timestep restriction

1) CFL-condition (advection): $\Delta t < \min\left(\frac{\Delta x}{v_x}, \frac{\Delta y}{v_y}\right)$
2) Neumann-condition (diffusion): $\Delta t < \min\left(\frac{Ra\,\Delta x^2}{2}, \frac{Ra\,\Delta y^2}{2}\right)$

For high Ra the CFL condition is typically limiting.

!! Always check both conditions.

Some notes on time step selection:

1) Advancing ADE with know v is a linear problem
   ⇒ expect to have a robut max Δt criterion = (cfl-number ~ 0.9)
2) CFL & Neumann conditions are applied "direction by direction" and seperately
   ⇒ will not give a robust criterion, because different directions and processes can amplify each other.

Right it seams we need a much to restrictive cfl-number and it seems to change with grid resolution.
= need better criterium.