Newton-Raphson Method

```
close all, clear all, clc
set_defaults()
```

To find the *root* of the non-linear algebraic system of N equations `r(u) = 0`. To do so we linearize the the system of equation by expanding it in a Taylor series and find the root of the linearized system to obtain an updated solution. Given an iterate, $\mathbf{u}^k$, we linearize the solution around it

$$\mathbf{r}(\mathbf{u}^{k+1}) \approx \mathbf{r}(\mathbf{u}^k) + \delta\mathbf{u}^k \frac{\mathrm{d}\mathbf{r}(\mathbf{u})}{\mathrm{d}\mathbf{u}}\bigg|_{\mathbf{u}^k} = 0,$$

so that the root of the linearized problem, $\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\mathbf{u}^k$, is the updated iterate. Here the derivative of the residual is generally known as the Jacobian matrix

$$\mathbf{J}(\mathbf{u}) \equiv \frac{\mathrm{d}\mathbf{r}(\mathbf{u})}{\mathrm{d}\mathbf{u}}$$

which is an N by N matrix of the N residuals in each cell with respect to the N unknowns in each cell. In general, computing **`J`** is the main challenge in implementing the Newton-Raphson method - in practice.

Given the residual and the Jacobian for an iterate, u, we can solve the following linear system for the update

`J(u)*du = -r(u)`

and compute the new iterate as

`u = u + du.`

The iteration is initialized with **`u = uold`** and terminated when both $|\mathbf{r}(\mathbf{u})| < \varepsilon$ and $|\mathbf{du}| < \varepsilon$ or the number of iterations exceeds a maximum. The latter is important, because the iteration is not guaranteed to converge! If the the initial guess is in the *basin of convergence* of the Newton-Raphson method, then it converges quardatically. Therefore the initial guess is critical to the convergence of the Newton-Raphson method.

Simple scalar example

Find the zero of the following non-linear function

$$r(x) = e^x - 2$$

```
r = @(x) exp(x)-2;
xplot = linspace(0,1.5,100);
plot(xplot,r(xplot),[0 1.5],[0 0],':'), hold on
plot(log(2),0,'o','markerfacecolor','w')
```

To solve this with the Newton-Raphson method, we need the Jacobian, $J = \mathrm{d}r/\mathrm{d}x = \exp(x)$, an intial guess, $x_0 = 1$, a convergence tolerance $\varepsilon = 10^{-6}$ and a maximum number of iterations Niter = 6;.

```
x0 = 3.5;
tol = 1e-6;
```

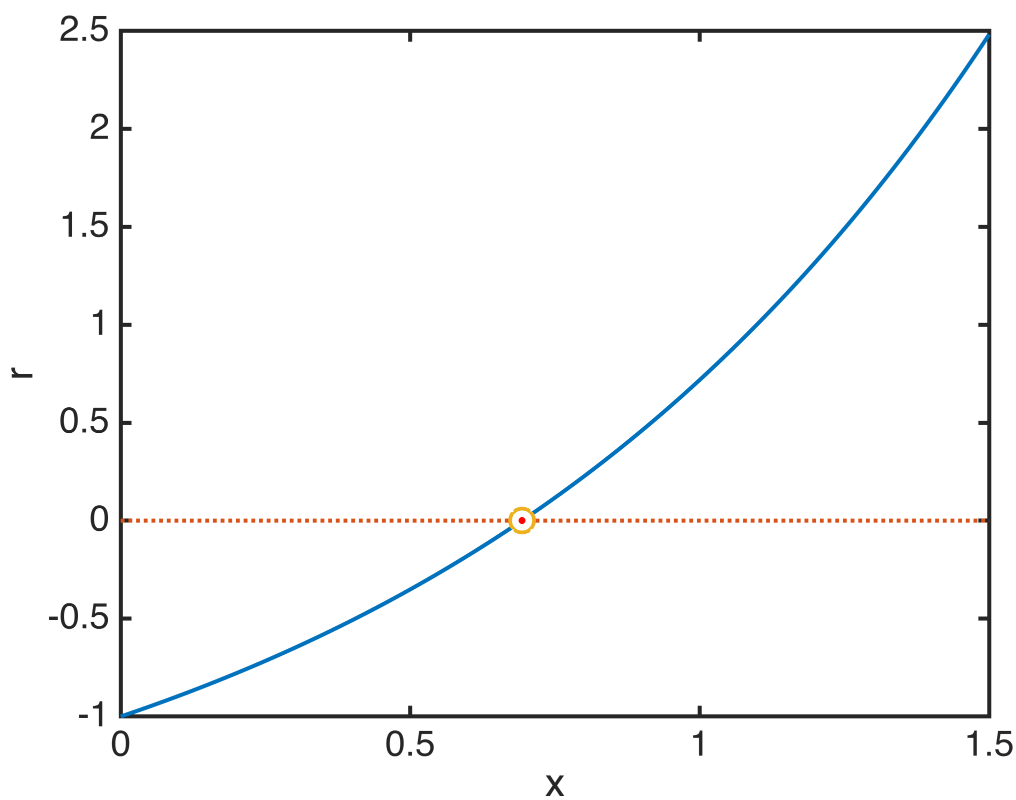
```
Niter = 10;
J = @(x) exp(x);
i = 0; x = x0; dx = 1;
fprintf('Newton-Raphson iterations:\n')
```

```
Newton-Raphson iterations:
```

```
while (norm(r(x)) > tol || norm(dx) > tol) && i <= Niter
    dx = -r(x)/J(x);
    x = x + dx;
    i = i + 1;
    fprintf('it = %d: r = %3.2e  dx = %3.2e\n',i,norm(r(x)),norm(dx))
end
```

```
it = 1: r = 1.09e+01  dx = 9.40e-01
it = 2: r = 3.56e+00  dx = 8.45e-01
it = 3: r = 9.30e-01  dx = 6.40e-01
it = 4: r = 1.33e-01  dx = 3.17e-01
it = 5: r = 4.07e-03  dx = 6.24e-02
it = 6: r = 4.12e-06  dx = 2.03e-03
it = 7: r = 4.25e-12  dx = 2.06e-06
it = 8: r = 0.00e+00  dx = 2.12e-12
```

```
plot(x,0,'r.')
xlabel 'x', ylabel 'r'
```



For a simple function like this with a single root, the Newton-Raphson method converges without problem. Note that the residual and update decrease by two orders of magnitude in the final few iterations, an indication that the convergence is quadratic.

Go show the script demo_scalar_newton.m

Simple 5 by 5 system example

Consider the 5 non-linear algebraic equations.

$$r_1(\mathbf{u}) = 3u_1 - 2\sqrt{u_3u_4}$$

$$r_2(\mathbf{u}) = \frac{u_2^2}{2} + u_5e^{u_3} + 5u_2$$

$$r_3(\mathbf{u}) = 7u_1^2u_3 + \pi u_4 + 2u_5^{\frac{1}{3}}$$

$$r_4(\mathbf{u}) = -\sqrt{u_2} + 3(u_1 - u_5)^2 + u_3u_4$$

$$r_5(\mathbf{u}) = u_1 - 4u_2 + 4u_5$$

We can create a system non-linear algebraic equations, $\mathbf{r}'$, with solution $\mathbf{u}^* = [1, 2, 3, 2, 1]^T$ by defining

$$\mathbf{r}'(\mathbf{u}) = \mathbf{r}(\mathbf{u}) - \mathbf{r}(\mathbf{u}^*).$$

Note that $\mathbf{u}^*$ is only one solution, there may be more!

```
% analytic solution
u_star = [1;2;3;2;1];

% residual function
r = @(u) [3*u(1)-2*sqrt(u(3).*u(4));...
          .5*u(2).^2+u(5).*exp(u(3))+5*u(2);...
          7*u(1).^2.*u(3)+pi*u(4)+2*u(5).^(1/3);...
          -sqrt(u(2))+3*(u(1)-u(5)).^2+u(3).*u(4);...
          u(1)-4*u(2)+4*u(5)];
r_prime = @(u) r(u)-r(u_star); % generates a residual with u_star as solution
```

The Jacobian is now a 5 by 5 matrix given by.

$$\mathbf{J} = \begin{bmatrix} 3 & 0 & \frac{-u_4}{\sqrt{u_3u_4}} & \frac{-u_3}{\sqrt{u_3u_4}} & 0 \\ 0 & u_2+5 & u_5e^{u_3} & 0 & e^{u_3} \\ 14u_1u_3 & 0 & 7u_1^2 & \pi & \frac{2}{3}u_5^{-\frac{2}{3}} \\ 6(u_1-u_5) & -\frac{1}{2\sqrt{u_2}} & u_4 & u_3 & -6(u_1-u_5) \\ 1 & -4 & 0 & 0 & 4 \end{bmatrix}$$

```
J = @(u) [3,0,-u(4)./sqrt(u(3).*u(4)),-u(3)./sqrt(u(3).*u(4)),0;...
          0,u(2)+5,u(5).*exp(u(3)),0,exp(u(3));...
          14*u(1).*u(3),0,7*u(1).^2,pi,2/3*u(5).^(-2/3);...
          6*(u(1)-u(5)),-1/2/sqrt(u(2)),u(4),u(3),-6*(u(1)-u(5));...
```

```
            1,-4,0,0,4];
```

This can now be solved for different initial guesses

```
u = 2*[1;1;1;1;1]; % initial guess - converges
% u = [1;1;1;1;1]; % does not converge (blows up)
% u = [2;3;1.5;.6;1]; % requires 28 iterations
Jac = J(u)
```

```
Jac = 5×5
    3.0000         0   -1.0000   -1.0000         0
         0    7.0000   14.7781         0    7.3891
   56.0000         0   28.0000    3.1416    0.4200
         0   -0.3536    2.0000    2.0000         0
    1.0000   -4.0000         0         0    4.0000
```

Here **J** is essentially a *full* matrix.

```
tol = 1e-8;
imax = 30;

fprintf('\nSolving 5x5 system with Newton-Raphson:');
```

```
Solving 5x5 system with Newton-Raphson:
```

```
i = 1; nres = 1; ndu = 1; % initialize Newton
while (nres > tol || ndu > tol) && i <= imax
    du = - J(u)\r_prime(u); ndu = norm(du);
    u = u+du; nres = norm(r_prime(u));
    fprintf('%d: res = %3.2e, du = %3.2e;\n',i,nres,ndu)
    if (nres < tol && ndu < tol) && i < imax
        fprintf('\nNewton iteration converged to tol = %3.2e\n',tol)
    elseif (nres > tol || ndu > tol) && i >= imax
        fprintf('\nNewton iteration did NOT converge to tol = %3.2e in %d
iterations!\n',tol,i)
    end
    i = i+1;
end
```

```
1: res = 1.35e+00, du = 1.44e+00;
2: res = 1.60e-01, du = 5.53e-01;
3: res = 3.60e-02, du = 1.43e-01;
4: res = 2.74e-03, du = 4.48e-02;
5: res = 1.76e-05, du = 3.48e-03;
6: res = 6.78e-10, du = 2.16e-05;
7: res = 7.17e-15, du = 8.35e-10;
Newton iteration converged to tol = 1.00e-08
```

```
u
```

```
u = 5×1
    1.0000
    2.0000
```

```
3.0000
2.0000
1.0000
```

Auxillary functions

set_defaults()

```
function [] = set_defaults()
set(0, ...
    'defaultaxesfontsize',   18, ...
    'defaultaxeslinewidth',   2.0, ...
    'defaultlinelinewidth',   2.0, ...
    'defaultpatchlinewidth',  2.0,...
    'DefaultLineMarkerSize', 12.0);
end
```