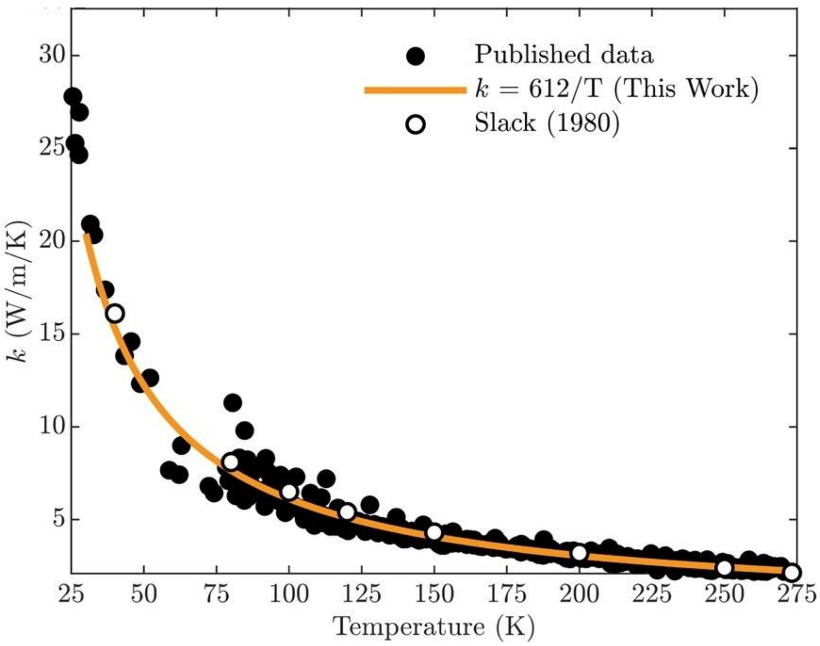
Non-linear heat conduction in Europa's ice shell

```
clear, close all, clc
set_defaults()
```

The thermal conductivity of ice is highly temperature dependent and the ice shels of icy ocean worlds experience large temperature differences, from 100 K at the surface to 273 K at the ice ocean interface.



(Carnahan and Wolfenbarger et al. 2021)

```
kappa0 = 612;
kappa = @(T) kappa0./T;
```

The steady-state temperature profile of the ice shell is therefore determined by the following non-linear heat conduction problem:

PDE: $-\nabla \cdot [\kappa(T)\nabla T] = 0$ on $z \in [0, H]$

BCs: $T(0) = T_o$ and $T(H) = T_s$

Here the thichness of the ice shell is $H = 30$ km, the ocean and surface temperatures are $T_o = 273$ K and $T_s = 100$ K, respectively. The dependence of the thermal conductivity on the temperature is $\kappa = \kappa_0/T$, where $\kappa_0 = 612$.

In 1D this problem has the following analytic solution:

$$T(z) = T_o\, e^{\ln(T_s/T_o)\, z/H}$$

$$q(z) = -\frac{\kappa_0}{H}\ln(T_s/T_o) = \text{constant.}$$

For the parameters given this leads to an increase of the thermal conductivity from 2.24 at the base to 6.12 W/(m K) at the top (a factor of 2.7).

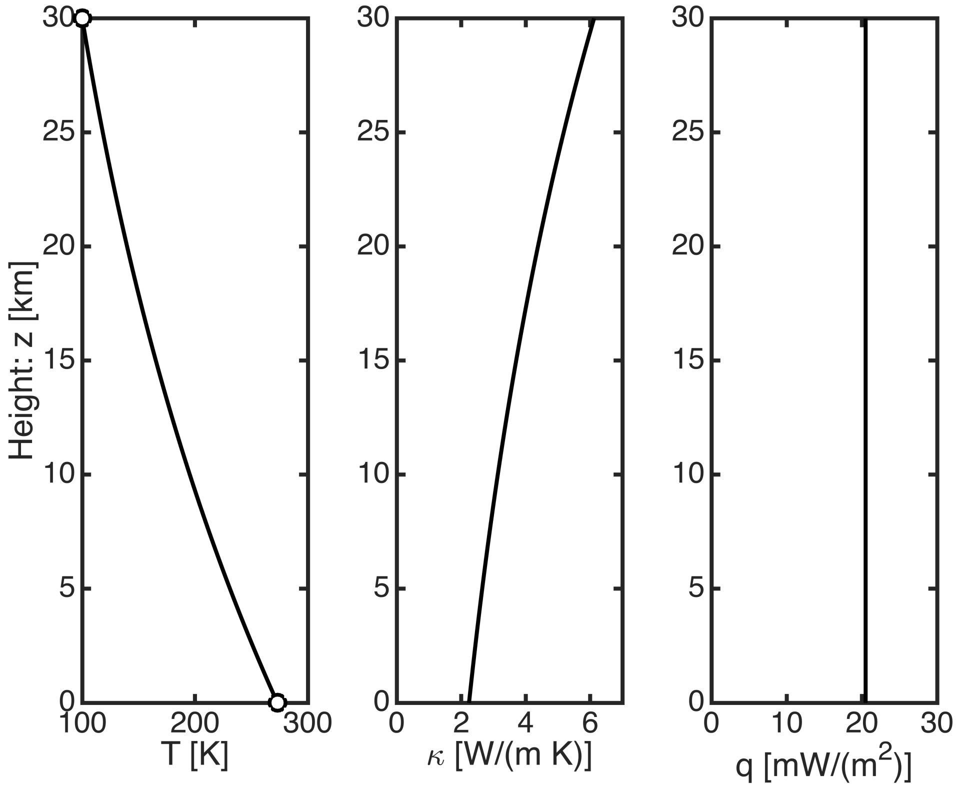
```
H = 30e3; % [m] ice shell thickness
Ts = 100; % [K] surface temperature
To = 273; % [K] ocean temperature
z_ana = linspace(0,H,1e2);

T_ana = @(z) To*exp(log(Ts/To)*z/H);
q_ana = @(z) -kappa0/H*log(Ts/To)+0*z;

subplot 131
plot(T_ana(z_ana),z_ana/1e3,'k-'), hold on
plot([To Ts],[0 H]/1e3,'ko','MarkerFaceColor','w')
xlabel('T [K]')
ylabel('Height: z [km]')

subplot 132
plot(kappa(T_ana(z_ana)),z_ana/1e3,'k-')
xlabel('\kappa [W/(m K)]')
set(gca,'xtick',[0:2:6])
xlim([0 7])

subplot 133
plot(q_ana(T_ana(z_ana))*1e3,z_ana/1e3,'k-'), hold on
xlabel('q [mW/(m^2)]')
xlim([0 30])
```



Newton-Rasphson with numerical Jacobian

```
Grid.xim = 0; Grid.xmax = H; Grid.Nx = 2e1;
Grid = build_grid(Grid);
[D,G,~,I,M] = build_ops(Grid);
fs = spalloc(Grid.N,1,0);
Kd = @(u) comp_mean(kappa(u),M,1,Grid,1);
res = @(u) -D*Kd(u)*G*u - fs;

BC.dof_dir = [Grid.dof_xmin;Grid.dof_xmax];
BC.dof_f_dir = [Grid.dof_f_xmin;Grid.dof_f_xmax];
BC.g = T_ana(Grid.xc(BC.dof_dir));

[B,N,fn,BC] = build_bnd(BC,Grid,I);

%% Newton iteration
tol = 1e-6;          % convergence tolerance
epsilon = 1e-8;      % perturbation for numerical derivative
nmax = 10;           % maximum number of iterations
nres = 1; ndu = 1;

u = (To+Ts)/2*ones(Grid.N,1);

% BC in NR are added to initial guess -> homogeneous during iteration
u(BC.dof_dir) = BC.g;
```

```
BC.g = [0;0];     % BC's for update du are homogeneous, because we initial
guess has correct BC.

fprintf('Newton-Raphson iteration\n')
```

```
Newton-Raphson iteration
```

```
n = 0;
while (nres > tol || ndu > tol) && n < nmax
     n = n+1;

    % 1) update solution
    Jac = comp_jacobian_steady_heat_flow_num(res,u,Grid,epsilon);
    du = solve_lbvp(Jac,-res(u),B,BC.g,N);
    u = u+du;

    % 2) check for convergence
    nres = norm(N'*res(u));
    ndu = norm(du);
    fprintf('it = %d: nres = %3.2e  ndu = %3.2e\n',n,nres,ndu)
end
```

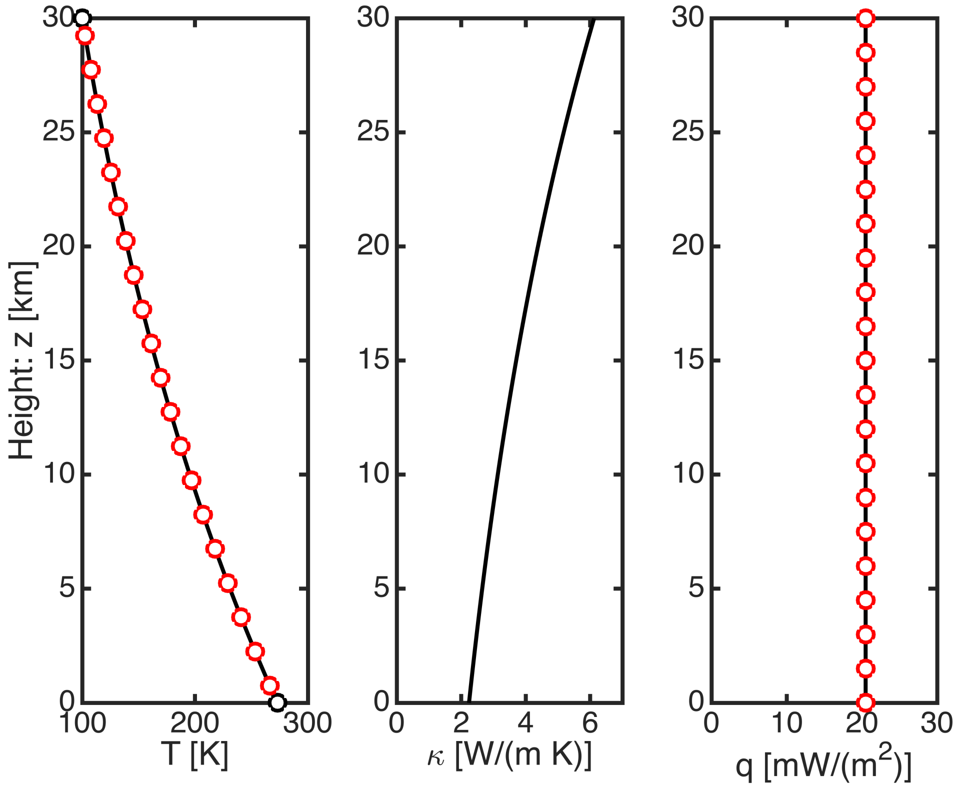
```
it = 1: nres = 5.08e-05  ndu = 2.07e+02
it = 2: nres = 3.66e-06  ndu = 2.44e+01
it = 3: nres = 1.94e-08  ndu = 1.21e+00
it = 4: nres = 4.20e-13  ndu = 4.41e-03
it = 5: nres = 7.27e-19  ndu = 7.82e-08
```

```

% Compute heat flux
q = comp_flux(D,Kd(u),G,u,fs,Grid,BC);


%% Plot numerical solution
subplot 131
plot(u,Grid.xc/1e3,'ro','MarkerFaceColor','w')

subplot 133
plot(q*1e3,Grid.xf/1e3,'ro','MarkerFaceColor','w')
```



Auxillary functions

Jacobian

```
function [Jac] = comp_jacobian_steady_heat_flow_num(res,u,Grid,epsilon)
u_perturb=u;
% Pre-allocate storage (important to speed up the for loop)
Jac = spalloc(Grid.N,Grid.N,3*Grid.N);

% %% Computing the Jacobian by finite difference
% % Loop over each unknown and
% % 1) Perturb it
% % 2) Compute the change in residual vector
% % 3) Store the change from unperturbed vector as column of Jacobnian
% % 4) Reset the perturbed value

for i=1:Grid.N
    u_perturb(i)=u(i)+epsilon;
    Jac(:,i)=(res(u_perturb)-res(u))/epsilon;
    u_perturb(i)=u(i);
end
```

```
end
```

Auxillary functions

set_defaults()

```
function [] = set_defaults()
    set(0, ...
    'defaultaxesfontsize',   18, ...
    'defaultaxeslinewidth',   2.0, ...
    'defaultlinelinewidth',   2.0, ...
    'defaultpatchlinewidth',  2.0,...
    'DefaultLineMarkerSize', 8.0);
end
```