Analytical Jacobian for steady non-linear heat flow

```
clear, close all, clc
```

The steady-state temperature profile of the ice shell is therefore determined by the following non-linear heat conduction problem:

PDE: $-\nabla \cdot [\kappa(T)\nabla T] = 0$ on $z \in [0, H]$

BCs: $T(0) = T_o$ and $T(H) = T_s$

Here the thichness of the ice shell is $H = 30$ km, the ocean and surface temperatures are $T_o = 273$ K and $T_s = 100$ K, respectively. The dependence of the thermal conductivity on the temperature is $\kappa = \kappa_0/T$, where $\kappa_0 = 612$.

In 1D this problem has the following analytic solution:

$$T(z) = T_o\, e^{\ln(T_s/T_o)\, z/H}$$

$$q(z) = -\frac{\kappa_0}{H}\ln(T_s/T_o) = \text{constant.}$$

For the parameters given this leads to an increase of the thermal conductivity from 2.24 at the base to 6.12 W/(m K) at the top (a factor of 2.7).

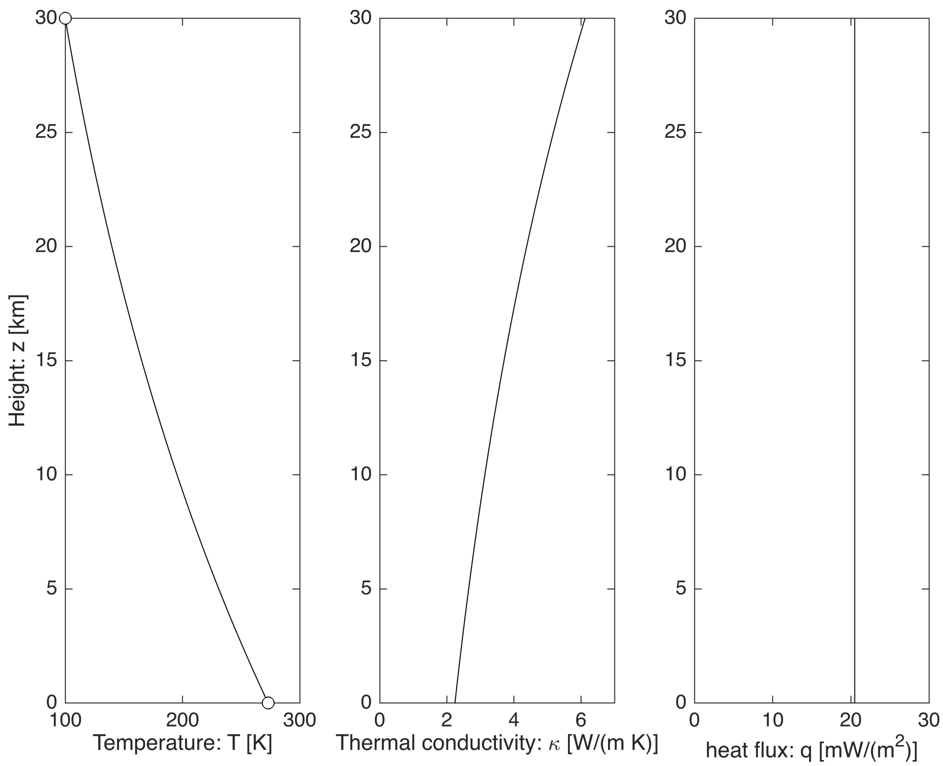
```
kappa0 = 612;
kappa = @(T) kappa0./T;
H = 30e3; % [m] ice shell thickness
Ts = 100; % [K] surface temperature
To = 273; % [K] ocean temperature
z_ana = linspace(0,H,1e2);

T_ana = @(z) To*exp(log(Ts/To)*z/H);
q_ana = @(z) -kappa0/H*log(Ts/To)+0*z;

subplot 131
plot(T_ana(z_ana),z_ana/1e3,'k-'), hold on
plot([To Ts],[0 H]/1e3,'ko','MarkerFaceColor','w')
xlabel('Temperature: T [K]')
ylabel('Height: z [km]')

subplot 132
plot(kappa(T_ana(z_ana)),z_ana/1e3,'k-')
xlabel('Thermal conductivity: \kappa [W/(m K)]')
xlim([0 7])

subplot 133
plot(q_ana(T_ana(z_ana))*1e3,z_ana/1e3,'k-'), hold on
xlabel('heat flux: q [mW/(m^2)]')
xlim([0 30])
```



Analytic Jacobian

To form the analyticq Jacobian of the discrete system we need a grid

```
Grid.xim = 0; Grid.xmax = H; Grid.Nx = 2e1;
Grid = build_grid(Grid);
[D,G,~,I,M] = build_ops(Grid);
fs = spalloc(Grid.N,1,0);
```

and the derivative of the non-linear thermal conductivity

$$\frac{\mathrm{d}\kappa}{\mathrm{d}T} = -\frac{\kappa_0}{T^2}$$

```
dkappa = @(u) -kappa0./(u.^2);
```

The residual is given by

r(u) = -D[Kd(u) G u] - fs

```
Kd = @(u) comp_mean(kappa(u),M,1,Grid,1); % arithmetic average
res = @(u) -D*Kd(u)*G*u - fs;
```

The the Jacobian is given by

J(u) = -D [GU dKd(u) + Kd(u) G]

where

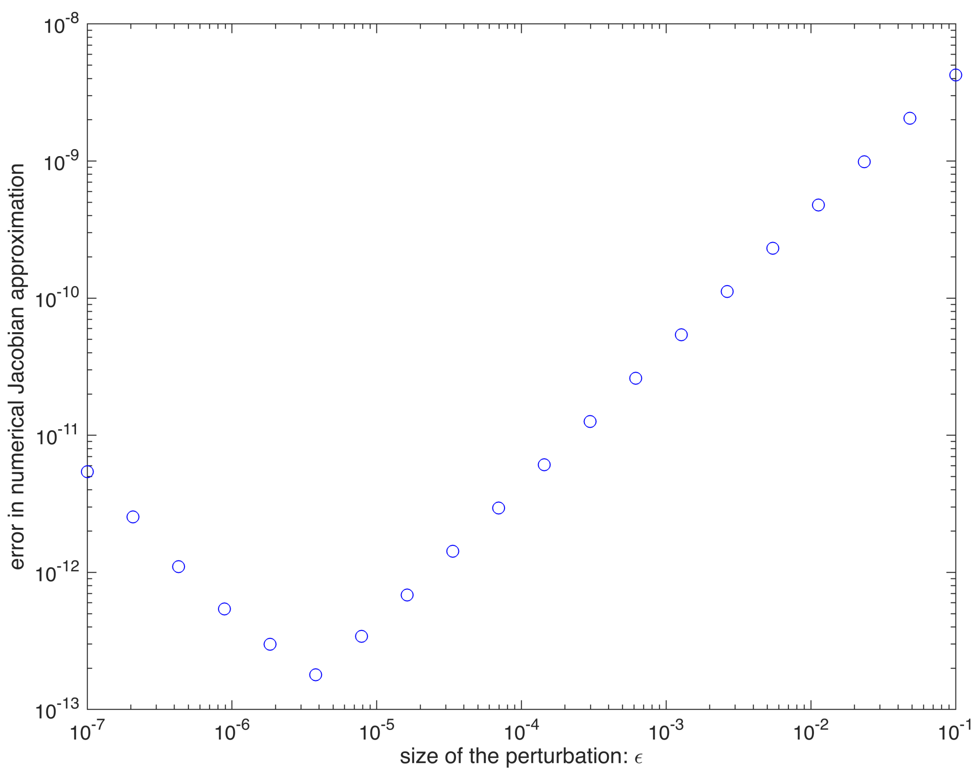
```
GU = @(u) spdiags(G*u,0,Grid.Nfx,Grid.Nfx);
dKd =@(u) M*spdiags(dkappa(u),0,Grid.Nx,Grid.Nx);
Jac = @(u) -D*( GU(u)*dKd(u) + Kd(u)*G );
```

Check against numerical Jacobian

```
% base state
u_test = (To+Ts)/2+(To-Ts)/10*sin(2*pi*Grid.xc/H);

% Compute analytical Jacobian
Jac_ana = Jac(u_test);

% Compute numerical Jacobian with decreasing epsilon
N_eps = 20;
eps_vec = logspace(-7,-1,N_eps);
for i = 1:N_eps
    Jac_num =
comp_jacobian_steady_heat_flow_num(res,u_test,Grid,eps_vec(i));
    Jerr(i) = norm(Jac_num(:)-Jac_ana(:));
end
figure
loglog(eps_vec,Jerr,'bo')
xlabel 'size of the perturbation: \epsilon'
ylabel 'error in numerical Jacobian approximation'
```



From this comparison we learn two things:

1. The analytical jacobian in derived and implemented correctly, because the numerical approximation converges to is with refinement.
2. The numerical approximation diverge for $\epsilon < 10^{-6}$, due to the relatively strong nonlinearity of $\kappa = \kappa_0/T$!

Note that the magnitude of ϵ when the numerical Jacobian diverges depends on the particular non-linearity of the conductivity. As such it cannot be know a-priory without comparison with the analytic Jaobian. Hence it is generally better to use the analytical Jacobian!

Solving the non-linear steady heat condution with Newton-Raphson method

```
BC.dof_dir = [Grid.dof_xmin;Grid.dof_xmax];
BC.dof_f_dir = [Grid.dof_f_xmin;Grid.dof_f_xmax];
BC.g = T_ana(Grid.xc(BC.dof_dir));

[B,N,fn,BC] = build_bnd(BC,Grid,I);

%% Newton iteration
tol = 1e-6;          % convergence tolerance
nmax = 10;           % maximum number of iterations


u = (To+Ts)/2*ones(Grid.N,1);

% BC in NR are added to initial guess -> homogneous during iteration
u(BC.dof_dir) = BC.g;
BC.g = [0;0];

nres = 1; ndu = 1; n = 0;
while (nres > tol || ndu > tol) && n < nmax
    % 1) Solve for update
    du = solve_lbvp(Jac(u),-res(u),B,BC.g,N);
    u = u+du;

    % 2) check for convergence
    nres = norm(N'*res(u));
    ndu = norm(du);
    n = n+1;
    fprintf('it = %d: nres = %3.2e  ndu = %3.2e\n',n,nres,ndu)
end
```

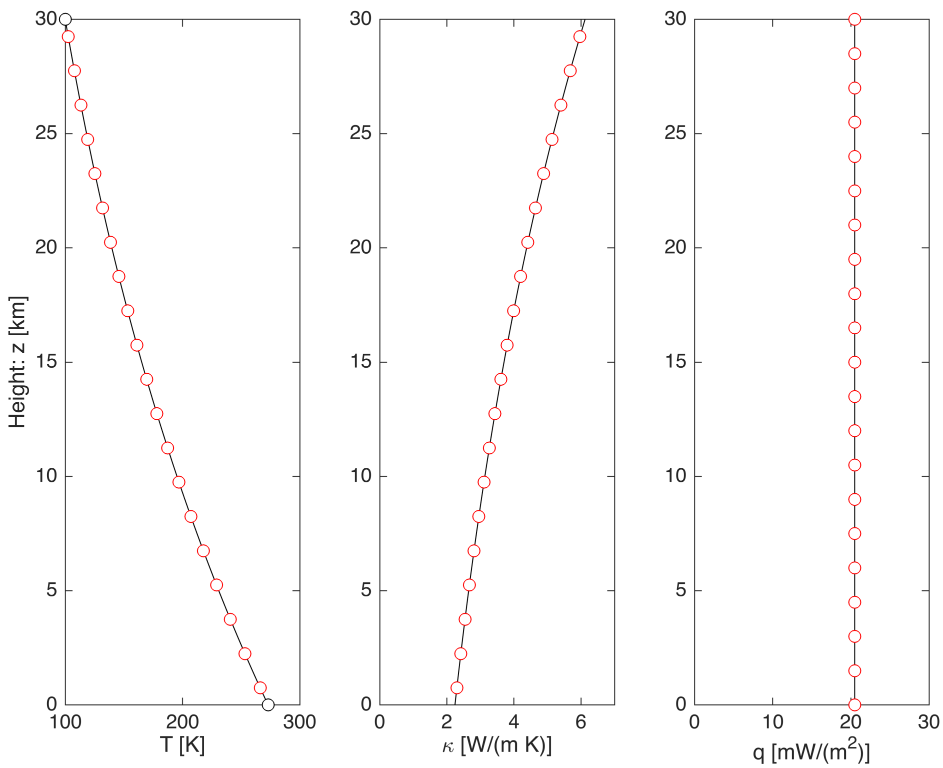
```
it = 1: nres = 5.08e-05  ndu = 2.07e+02
it = 2: nres = 3.66e-06  ndu = 2.44e+01
it = 3: nres = 1.94e-08  ndu = 1.21e+00
it = 4: nres = 4.37e-13  ndu = 4.41e-03
it = 5: nres = 2.76e-19  ndu = 8.03e-08
```

```
% Compute heat flux
q = comp_flux(D,Kd(u),G,u,fs,Grid,BC);


figure
subplot 131
plot(T_ana(z_ana),z_ana/1e3,'k-'), hold on
plot(u,Grid.xc/1e3,'ro','MarkerFaceColor','w')
plot([To Ts],[0 H]/1e3,'ko','MarkerFaceColor','w')
xlabel('T [K]')
ylabel('Height: z [km]')

subplot 132
plot(kappa(T_ana(z_ana)),z_ana/1e3,'k-'), hold on
plot(kappa(u),Grid.xc/1e3,'ro','MarkerFaceColor','w')
xlabel('\kappa [W/(m K)]')
xlim([0 7])

subplot 133
plot(q_ana(T_ana(z_ana))*1e3,z_ana/1e3,'k-'), hold on
plot(q*1e3,Grid.xf/1e3,'ro','MarkerFaceColor','w')
xlabel('q [mW/(m^2)]')
xlim([0 30])
```



Auxillary functions

Jacobian

```
function [Jac] = comp_jacobian_steady_heat_flow_num(res,u,Grid,epsilon)
u_perturb=u;
% Pre-allocate storage (important to speed up the for loop)
Jac = spalloc(Grid.N,Grid.N,3*Grid.N);

% %% Computing the Jacobian by finite difference
% % Loop over each unknown and
% % 1) Perturb it
% % 2) Compute the change in residual vector
% % 3) Store the change from unperturbed vector as column of Jacobnian
% % 4) Reset the perturbed value

for i=1:Grid.N
    u_perturb(i)=u(i)+epsilon;
    Jac(:,i)=(res(u_perturb)-res(u))/epsilon;
    u_perturb(i)=u(i);
end
end
```